

# PRICE'S LAW ON NONSTATIONARY SPACETIMES

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ABSTRACT. In this article we study the pointwise decay properties of solutions to the wave equation on a class of nonstationary asymptotically flat backgrounds in three space dimensions. Under the assumption that uniform energy bounds and a weak form of local energy decay hold forward in time we establish a  $t^{-3}$  local uniform decay rate (Price's law [54]) for linear waves. As a corollary, we also prove Price's law for certain small perturbations of the Kerr metric.

This result was previously established by the second author in [64] on stationary backgrounds. The present work was motivated by the problem of nonlinear stability of the Kerr/Schwarzschild solutions for the vacuum Einstein equations, which seems to require a more robust approach to proving linear decay estimates.

## 1. INTRODUCTION

In this article we consider the question of pointwise decay for solutions to the wave equation on certain asymptotically flat backgrounds. Our interest in this problem originates in general relativity, more precisely the wave equation on Schwarzschild and Kerr backgrounds. There the expected local decay, heuristically derived by Price [54] in the Schwarzschild case, is  $t^{-3}$ . This conjecture was considered independently in two recent articles [26] and [64].

The work in [26] is devoted to the Schwarzschild space-time, where separation of variables can be used; in that context, very precise sharp local decay bounds are established for each of the spherical modes. Precisely, it is shown that the  $k$ -th mode leads to  $t^{-3-2k}$  local decay.

The work in [64], on the other hand, applies to a large class of stationary asymptotically flat space-times, and asserts that if local energy decay holds then Price's law holds. Further, sharp pointwise decay rates are established in the full forward light cone; these have the form  $t^{-1}\langle t-r \rangle^{-2}$ . Local energy decay (described later in the paper) is known to hold in Schwarzschild and Kerr space-times.

Both of the above results involve taking the Fourier transform in time and hence rely heavily on the stationarity assumption. The aim of this article is to prove the same result as in [64], namely that local energy decay implies Price's law, but without the stationarity assumption. The proof below is more robust than the one in [64], and improves on the classical vector field method. As an application, in the last section of the paper we prove that local energy decay (and thus Price's law) holds for a class of nonstationary perturbations of the Kerr space-time.

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Just as in [64], this work is based on the idea that the local energy estimates contain all the important local information concerning the flow, and that only leaves the analysis near spatial infinity to be understood. In the context of asymptotically flat metrics, this idea originates in earlier work [65] and [48] where it is proved that local energy decay implies Strichartz estimates in the asymptotically flat setting, first for the Schrödinger equation and then for the wave equation. The same principle was exploited in [45] and [67] to prove Strichartz estimates for the wave equation on the Schwarzschild and then on the Kerr space-time.

**1.1. Notations and the regularity of the metric.** We use  $(t = x_0, x)$  for the coordinates in  $\mathbb{R}^{1+3}$ . We use Latin indices  $i, j = 1, 2, 3$  for spatial summation and Greek indices  $\alpha, \beta = 0, 1, 2, 3$  for space-time summation. In  $\mathbb{R}^3$  we also use polar coordinates  $x = r\omega$  with  $\omega \in \mathbb{S}^2$ . By  $\langle r \rangle$  we denote a smooth radial function which agrees with  $r$  for large  $r$  and satisfies  $\langle r \rangle \geq 2$ . We consider a partition of  $\mathbb{R}^3$  into the dyadic sets  $A_R = \{\langle r \rangle \approx R\}$  for  $R \geq 1$ , with the obvious change for  $R = 1$ .

To describe the regularity of the coefficients of the metric, we use the following sets of vector fields:

$$T = \{\partial_t, \partial_i\}, \quad \Omega = \{x_i \partial_j - x_j \partial_i\}, \quad S = t \partial_t + x \partial_x,$$

namely the generators of translations, rotations and scaling. We set  $Z = \{T, \Omega, S\}$ . Then we define the classes  $S^Z(r^k)$  of functions in  $\mathbb{R}^+ \times \mathbb{R}^3$  by

$$a \in S^Z(r^k) \iff |Z^j a(t, x)| \leq c_j \langle r \rangle^k, \quad j \geq 0.$$

By  $S_{rad}^Z(r^k)$  we denote spherically symmetric functions in  $S^Z(r^k)$ .

The estimates in this article apply to solutions for an inhomogeneous problem of the form

$$(1.1) \quad (\square_g + V)u = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

associated to d'Alembertian  $\square_g$  corresponding to a Lorentzian metric  $g$ , a potential  $V$ , nonhomogeneous term  $f$  and compactly supported initial data  $u_0, u_1$ . For the metric  $g$  we consider two cases:

**Case A:**  $g$  is a smooth Lorentzian metric in  $\mathbb{R}^+ \times \mathbb{R}^3$ , with the following properties:

- (i) The level sets  $t = \text{const}$  are space-like.
- (ii)  $g$  is asymptotically flat in the following sense:

$$g = m + g_{sr} + g_{lr},$$

where  $m$  stands for the Minkowski metric,  $g_{lr}$  is a stationary long range spherically symmetric component, with  $S_{rad}^Z(r^{-1})$  coefficients, of the form

$$g_{lr} = g_{lr,tt}(r)dt^2 + g_{lr,tr}(r)dtdr + g_{lr,rr}(r)dr^2 + g_{lr,\omega\omega}(r)r^2d\omega^2$$

and  $g_{sr}$  is a short range component of the form

$$g_{sr} = g_{sr,tt}dt^2 + 2g_{sr,ti}dtdx_i + g_{sr,ij}dx_i dx_j$$

with  $S^Z(r^{-2})$  coefficients.

We remark that these assumptions guarantee that  $\partial_t$  is time-like near spatial infinity, but not necessarily in a compact set. This leads us to the second case we consider:

**Case B:**  $g$  is a smooth Lorentzian metric in an exterior domain  $\mathbb{R} \times \mathbb{R}^3 \setminus B(0, R_0)$  which satisfies (i),(ii) above in its domain, and in addition  
 (iii) the lateral boundary  $\mathbb{R} \times \partial B(0, R_0)$  is outgoing space-like.

This latter condition insures that the corresponding wave equation is well-posed forward in time. This assumption is satisfied in the case of the Schwarzschild and Kerr metrics (or small perturbations thereof) in suitable advanced time coordinates. There the parameter  $R_0$  is chosen so that  $0 < R_0 < 2M$  in the case of the Schwarzschild metric, respectively  $r^- < R_0 < r^+$  in the case of Kerr (see [30] for the definition of  $r^\pm$ ), so that the exterior of the  $R_0$  ball contains a neighbourhood of the event horizon.

**1.2. Normalized coordinates.** Our decay results are expressed relative to the distance to the Minkowski null cone  $\{t = |x|\}$ . This can only be done provided that there is a null cone associated to the metric  $g$  which is within  $O(1)$  of the Minkowski null cone. However, in general the long range component of the metric produces a logarithmic correction to the cone. This issue can be remedied via a change of coordinates which roughly corresponds to using Regge-Wheeler coordinates in Schwarzschild/Kerr near spatial infinity. See [64]. After a further conformal transformation<sup>1</sup>, see also [64], the metric  $g$  is reduced to a normal form where

$$g_{lr} = g_\omega(r)r^2 d\omega^2, \quad g_\omega \in S_{rad}^Z(r^{-1}).$$

In particular, we can replace  $\square_g$  by an operator of the form

$$(1.2) \quad P = \square + Q$$

where  $\square$  denotes the d'Alembertian in the Minkowski metric and the perturbation  $Q$  has the form

$$(1.3) \quad Q = g^\omega \Delta_\omega + \partial_\alpha g_{sr}^{\alpha\beta} \partial_\beta + V, \quad g_{sr}^{\alpha\beta} \in S^Z(r^{-2}), \quad g^\omega \in S_{rad}^Z(r^{-3}), \quad V \in S^Z(r^{-3}).$$

We call these coordinates normal coordinates. All of the analysis in the paper is done in normal coordinates and with  $g$  in normal form. The full perturbation  $Q$  above has only short range effects.

**1.3. Uniform energy bounds.** The Cauchy data at time  $t$  for the evolution (1.1) is given by  $(u(t), \partial_t u(t))$ . To measure it we use the Sobolev spaces  $H^k$ , with the qualification that in Case A this means  $H^k := H^k(\mathbb{R}^3)$ , while in Case B we use the obvious modification  $H^k := H^k(\mathbb{R}^3 \setminus B(0, R_0))$ . We begin with the following definition:

**Definition 1.1.** *We say that the evolution (1.1) is forward bounded if the following estimates hold:*

$$(1.4) \quad \|\nabla u(t_1)\|_{H^k} \leq c_k(\|\nabla u(t_0)\|_{H^k} + \|f\|_{L^1([t_0, t_1]; H^k)}), \quad 0 \leq t_0 \leq t_1, \quad k \geq 0.$$

It suffices to have this property when  $f = 0$ . Then the  $f$  term can be added in by the Duhamel formula. One case when the uniform forward bounds above are easy to establish is when  $\partial_t$  is a Killing vector which is everywhere time-like and  $V$  is nonnegative and stationary. Otherwise, there is no general result, but various cases have been studied on a case by case basis.

We remark that in the case of the Schwarzschild and Kerr space-times  $\partial_t$  is not everywhere time-like, so the forward boundedness is not straightforward. However,

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<sup>1</sup>which changes the potential  $V$

it is known to hold for Schwarzschild (see [24] and [45]) as well as for Kerr with small angular momentum (see [29], [23], [66]) and for a class of small stationary perturbations of Schwarzschild (see [23]).

The forward boundedness is not explicitly used in what follows, but it is defined here since it is usually seen as a prerequisite for everything that follows.

**1.4. Local energy decay.** A stronger property of the wave flow is local energy decay. We introduce the local energy norm  $LE$

$$(1.5) \quad \begin{aligned} \|u\|_{LE} &= \sup_R \|\langle r \rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R} \times A_R)} \\ \|u\|_{LE[t_0, t_1]} &= \sup_R \|\langle r \rangle^{-\frac{1}{2}} u\|_{L^2([t_0, t_1] \times A_R)}, \end{aligned}$$

its  $H^1$  counterpart

$$(1.6) \quad \begin{aligned} \|u\|_{LE^1} &= \|\nabla u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE} \\ \|u\|_{LE^1[t_0, t_1]} &= \|\nabla u\|_{LE[t_0, t_1]} + \|\langle r \rangle^{-1} u\|_{LE[t_0, t_1]}, \end{aligned}$$

as well as the dual norm

$$(1.7) \quad \begin{aligned} \|f\|_{LE^*} &= \sum_R \|\langle r \rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R} \times A_R)} \\ \|f\|_{LE^*[t_0, t_1]} &= \sum_R \|\langle r \rangle^{\frac{1}{2}} f\|_{L^2([t_0, t_1] \times A_R)}. \end{aligned}$$

These definitions are specific to  $(1+3)$ -dimensional problems. Some appropriate modifications are needed in other dimensions, see for instance [48]. We also define similar norms for higher Sobolev regularity

$$\begin{aligned} \|u\|_{LE^{1,k}} &= \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{LE^1} \\ \|u\|_{LE^{1,k}[t_0, t_1]} &= \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{LE^1[t_0, t_1]} \\ \|u\|_{LE^{0,k}[t_0, t_1]} &= \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{LE[t_0, t_1]}, \end{aligned}$$

respectively

$$\begin{aligned} \|f\|_{LE^{*,k}} &= \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{LE^*} \\ \|f\|_{LE^{*,k}[t_0, t_1]} &= \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{LE^*[t_0, t_1]}. \end{aligned}$$

In Case A above this leads to the following

**Definition 1.2.** *We say that the evolution (1.1) has the local energy decay property if the following estimate holds:*

$$(1.8) \quad \|u\|_{LE^{1,k}[t_0, \infty)} \leq c_k (\|\nabla u(t_0)\|_{H^k} + \|f\|_{LE^{*,k}[t_0, \infty)}), \quad t_0 \geq 0, \quad k \geq 0$$

in  $\mathbb{R} \times \mathbb{R}^3$ .

The first local energy decay estimates for the wave equation were proved in the work of Morawetz [50], [51], [52]; estimates of this type are often called Morawetz estimates. There is by now an extensive literature devoted to this topic and its

applications; without being exhaustive we mention [59], [36], [56], [34], [35], [9], [46], [47], [58], [31].

The sharp form of the estimates as well as the notations above are from Metcalfe-Tataru [48]; this paper also contains a proof of the local energy decay estimates for small (time dependent) long range perturbations of the Minkowski space-time and further references. See also [49], [1], [46] for time dependent perturbations, as well as, e.g., [8], [7], [57] for time independent, nontrapping perturbations. There is a related family of local energy estimates for the Schrödinger equation. See, e.g., the original works [17], [55], [68] in this direction as well as [25], [18] in variable geometries. For notations and estimates most reminiscent to those used here, we refer the reader to [65] and [44].

In Case B an estimate such as (1.8) cannot hold due to the existence of trapped rays, i.e. null geodesics confined to a compact spatial region. However a weaker form of the local energy decay may still hold if the trapped null geodesics are hyperbolic. This is the case for both the Schwarzschild metric and for the Kerr metric with angular momentum  $|a| < M$ . To state such bounds we introduce a weaker version of the local energy decay norm

$$\begin{aligned} \|u\|_{LE_w^1} &= \|(1 - \chi)\nabla u\|_{LE} + \|\langle r \rangle^{-1}u\|_{LE} \\ \|u\|_{LE_w^1[t_0, t_1]} &= \|(1 - \chi)\nabla u\|_{LE[t_0, t_1]} + \|\langle r \rangle^{-1}u\|_{LE[t_0, t_1]} \end{aligned}$$

for some spatial cutoff function  $\chi$  which is smooth and compactly supported. Heuristically,  $\chi$  is chosen so that it equals 1 in a neighbourhood of the trapped set. We define as well a dual type norm

$$\begin{aligned} \|f\|_{LE_w^*} &= \|\chi f\|_{L^2 H^1} + \|(1 - \chi)f\|_{LE^*} \\ \|f\|_{LE_w^*[t_0, t_1]} &= \|\chi f\|_{L^2[t_0, t_1] H^1} + \|(1 - \chi)f\|_{LE^*[t_0, t_1]}. \end{aligned}$$

As before we define the higher norms  $LE_w^{1,k}$  respectively  $LE_w^{*,k}$ .

**Definition 1.3.** *We say that the evolution (1.1) has the weak local energy decay property if the following estimate holds:*

$$(1.9) \quad \|u\|_{LE_w^{1,k}[t_0, \infty)} \leq c_k(\|\nabla u(t_0)\|_{H^k} + \|f\|_{LE_w^{*,k}[t_0, \infty)}), \quad k \geq 0, t_0 \geq 0$$

*in either  $\mathbb{R} \times \mathbb{R}^3$  or in the exterior domain case.*

Note that this implies in particular

$$(1.10) \quad \|u\|_{LE^{1,k}[t_0, \infty)} \leq c_k(\|\nabla u(t_0)\|_{H^{k+1}} + \|f\|_{LE^{*,k+1}[t_0, \infty)}),$$

hence we can get rid of the loss at the trapped set by paying the price of one extra derivative.

Two examples where weak local energy decay is known to hold are the Schwarzschild space-time and the Kerr space-time with small angular momentum  $|a| \ll M$ . In the Schwarzschild case the above form of the local energy bounds with  $k = 0$  was obtained in [45], following earlier results in [41], [4], [5], [3], [6], [24], [22]. The number of derivatives lost in (1.10) can be improved to any  $\epsilon > 0$  (see, for example, [3], [45]), but that is not relevant for the problem at hand.

In the case of the Kerr space-time with small angular momentum  $|a| \ll M$  the local energy estimates were first proved in Tataru-Tohaneanu [66], in a form which is compatible with Definition 1.3. Stronger bounds near the trapped set as well as Strichartz estimates are contained in the paper of Tohaneanu [67]. For related work

we also refer the reader to [67], [20] and [2]. For large angular momentum  $|a| < M$  a similar estimate was proved in [19] for axisymmetric solutions.

The high frequency analysis of the dynamics near the hyperbolic trapped orbits is a very interesting related topic, but does not have much to do with the present article. For more information we refer the reader to [16], [14], [53], [70], [10].

One disadvantage of the bound (1.10) is that it is not very stable with respect to perturbations. To compensate for that, for the present result we need to introduce an additional local energy type bound:

**Definition 1.4.** *We say that the problem (1.1) satisfies stationary local energy decay bounds if on any time interval  $[t_0, t_1]$  and  $k \geq 0$  we have*

$$(1.11) \quad \|u\|_{LE^{1,k}[t_0, t_1]} \lesssim_k \|\nabla u(t_0)\|_{H^k} + \|\nabla u(t_1)\|_{H^k} + \|f\|_{LE^{*,k}[t_0, t_1]} + \|\partial_t u\|_{LE^{0,k}[t_0, t_1]}.$$

Unlike the weak local energy decay, here there is no loss of derivatives. Instead, the price we pay is the local energy of  $\partial_t u$  on the right. Heuristically, (1.11) can be viewed as a consequence of (1.9) whenever  $\partial_t$  is timelike near the trapped set. In the stationary case, this can be thought of as a substitute of an elliptic estimate at zero frequency.

While one can view the stationary local energy decay as a consequence of the local energy decay, it is in effect far more robust and easier to prove than the weak local energy decay provided that  $\partial_t$  is timelike near the trapped set. This allows one to completely sidestep all trapping related issues. This difference is quite apparent in the last section of the paper, where we separately establish both stationary local energy decay and weak local energy decay for small perturbations of Kerr. While the former requires merely smallness of the perturbation uniformly in time, the latter needs a much stronger  $t^{-1-}$  decay to Kerr.

Because of the above considerations, for our first (and main) result in the theorem below we are only using as hypothesis the stationary local energy condition.

**1.5. The main result.** Given a multiindex  $\alpha$  we denote by  $u_\alpha = Z^\alpha u$ . By  $u_{\leq m}$  we denote the collection of all  $u_\alpha$  with  $|\alpha| \leq m$ . We are now ready to state the main result of the paper:

**Theorem 1.5.** *Let  $g$  be a metric which satisfies the conditions (i), (ii) in  $\mathbb{R} \times \mathbb{R}^3$ , or (i), (ii), (iii) in  $\mathbb{R} \times \mathbb{R}^3 \setminus B(0, R_0)$ , and  $V$  belonging to  $S(r^{-3})$ . Assume that the evolution (1.1) satisfies the stationary local energy bounds from Definition 1.4. Suppose that in normalized coordinates the function  $u$  solves  $\square u = f$  and is supported in the forward cone  $C = \{t \geq r - R_1\}$  for some  $R_1 > 0$ . Then the following estimate holds in normalized coordinates for large enough  $m$ :*

$$(1.12) \quad |u(t, x)| \lesssim \kappa \frac{1}{\langle t \rangle \langle t - r \rangle^2}, \quad |\nabla u(t, x)| \lesssim \kappa \frac{1}{\langle r \rangle \langle t - r \rangle^3}$$

where

$$\kappa = \|u_{\leq m}\|_{LE^1} + \|t^{\frac{5}{2}} f_{\leq m}\|_{LE^*} + \|rt^{\frac{5}{2}} \nabla f_{\leq m}\|_{LE^*}.$$

*If in addition the weak local energy bounds (1.9) hold then the same result is valid for all forward solutions to (1.1) with data  $(u_0, u_1)$  and  $f$  supported inside the cone  $C$  and*

$$\kappa = \|\nabla u(0)\|_{H^m} + \|t^{\frac{5}{2}} f_{\leq m}\|_{LE^*} + \|rt^{\frac{5}{2}} \nabla f_{\leq m}\|_{LE^*}.$$

We remark that we actually prove a slightly stronger result, with  $\kappa$  replaced by  $C_4$  in Lemma 3.21.

As an application of this result, in the last part of the paper we prove Price's Law for certain small, time-dependent perturbations of Kerr spacetimes with small angular momentum (and  $V = 0$ ).

The problem of obtaining pointwise decay rates for linear and nonlinear wave equations has had a long history. Dispersive  $L^1 \rightarrow L^\infty$  estimates providing  $t^{-1}$  decay of  $3 + 1$  dimensional waves in the Minkowski setting have been known for a long time.

The need for weighted decay inside the cone arose in John's proof [32] of the Strauss conjecture in  $3 + 1$  dimensions. Decay bounds for  $\square + V$  with  $V = O(r^{-3})$ , similar to those given by Price's heuristics, were obtained by Strauss-Tsutaya [60] and Szpak [62], [61]. See also Szpak-Bizoń-Chmaj-Rostworowski [63].

A more robust way of proving pointwise estimates via  $L^2$  bounds and Sobolev inequalities was introduced in the work of Klainerman, who developed the so-called vector field method, see for instance [38], [37]. This idea turned out to have a myriad of applications and played a key role in the Christodoulou-Klainerman [15] proof of the asymptotic stability of the Minkowski space time for the vacuum Einstein equations.

In the context of the Schwarzschild space-time, Price was the first to heuristically compute the  $t^{-3}$  decay rate for linear waves. More precise heuristic computations were carried out later by Ching-Leung-Suen-Young [12], [13]. Following work of Wald [69], the first rigorous proof of the boundedness of the solutions to the wave equation was given in Kay-Wald [33].

Uniform pointwise  $t^{-1}$  decay estimates were obtained by Blue-Sterbenz [3] and also Dafermos-Rodnianski [24]; the bounds in the latter paper are stronger in that they extend uniformly up to the event horizon. A local  $t^{-\frac{3}{2}}$  decay result was obtained by Luk [43]. These results are obtained using multiplier techniques, related to Klainerman's vector field method; in particular the conformal multiplier plays a key role.

Another venue which was explored was to use the spherical symmetry in order to produce an expansion into spherical modes and to study the corresponding ode. This was pursued by Kronthaler [40], [39], who in the latter article was able to establish the sharp Price's Law in the spherically symmetric case. A related analysis was carried out later by Donniger-Schlag-Soffer [27] for all the spherical modes; they establish a  $t^{-2-2k}$  local decay for the  $k$ -th spherical mode. Later the same authors obtain the sharp  $t^{-3-2k}$  in [26].

Switching to Kerr, the first decay results there were obtained by Finster-Kamran-Smoller-Yau [28]. Later Dafermos-Rodnianski [20] and Anderson-Blue [2] were able to extend their Schwarzschild results to Kerr, obtaining almost a  $t^{-1}$  decay. This was improved to  $t^{-1}$  by Dafermos-Rodnianski [21] and to  $t^{-\frac{3}{2}}$  by Luk [42].

Finally, the  $t^{-3}$  decay result (Price's Law) was proved by the second author in [64]; the result there applies for a large class of stationary, asymptotically flat space-times. In addition, in [64] the optimal decay is obtained in the full forward light cone.

## 2. VECTOR FIELDS AND LOCAL ENERGY DECAY

The primary goal of this section is to develop localized energy estimates when the vector fields  $Z$  are applied to the solution  $u$ .

**2.1. Vector fields: notations and definitions.** For a triplet  $\alpha = (i, j, k)$  of multi-indices  $i, j$  and nonnegative integer  $k$  we denote  $|\alpha| = |i| + 3|j| + 9k$  and

$$u_\alpha = T^i \Omega^j S^k u.$$

On the family of such triplets we introduce the ordering induced by the ordering of integers. Namely, if  $\alpha_1 = (i_1, j_1, k_1)$  and  $\alpha_2 = (i_2, j_2, k_2)$  then we define

$$\alpha_1 \leq \alpha_2 \quad \equiv \quad |i_1| \leq |i_2|, \quad |j_1| \leq |j_2|, \quad k_1 \leq k_2.$$

We use  $<$  for the case when equality does not hold. For an integer  $m$  we denote

$$\alpha_1 \leq \alpha_2 + m \quad \equiv \quad \alpha_1 \leq \alpha_2 + \beta, \quad |\beta| \leq m.$$

We also define

$$\begin{aligned} u_{\leq m} &= (u_\alpha)_{|\alpha| \leq m} \\ u_{\leq \beta} &= (u_\alpha)_{\alpha \leq \beta}. \end{aligned}$$

and the analogues for  $<$  instead of  $\leq$ .

We now study the commutation properties of the vector fields with  $P$ . Denoting by  $\mathcal{Q}_{sr}$  the class of all operators of the form

$$\partial_\alpha g^{\alpha\beta} \partial_\beta + V, \quad g_{sr}^{\alpha\beta} \in S^Z(r^{-2}), \quad V \in S^Z(r^{-3}),$$

we see that  $Q$  defined in (1.3) consists of  $R\Omega^2$  where  $R \in S_{rad}^Z(r^{-3})$  plus an element of  $\mathcal{Q}_{sr}$ .

We now record the commutators of  $P$  with vector fields. The commutator with  $T$  yields

$$(2.13) \quad [P, T] = Q, \quad Q \in \mathcal{Q}_{sr}.$$

The same applies with  $\Omega$ ,

$$(2.14) \quad [P, \Omega] = Q, \quad Q \in \mathcal{Q}_{sr}.$$

However, in the case of  $S$  we get an extra contribution arising from the long range part of  $P$ ,

$$(2.15) \quad PS - (S + 2)P = Q + R\Omega^2, \quad Q \in \mathcal{Q}_{sr}, \quad R \in S_{rad}^Z(r^{-3}).$$

Further commutations preserve the  $\mathcal{Q}_{sr}$  class. Thus we can write the equation for  $\Omega^j S^k u$  in the form

$$(2.16) \quad P\Omega^j S^k u = \Omega^j (S + 2)^k P u + Q u_{\leq 3j+9k-3} \quad Q \in \mathcal{Q}_{sr}.$$

Suppose the function  $u$  solves the equation

$$Pu = f.$$

Commuting with all vector fields, we obtain equations for the functions  $u_\alpha$ . These can be written in the forms

$$(2.17) \quad Pu_\alpha = Qu_{<|\alpha|} + f_{\leq|\alpha|} =: F_\alpha, \quad Q \in \mathcal{Q}_{sr},$$

$$(2.18) \quad \square u_\alpha = Qu_{\leq|\alpha|} + R\Omega^2 u_{\leq|\alpha|} + f_{\leq|\alpha|} =: G_\alpha, \quad Q \in \mathcal{Q}_{sr}, \quad R \in S_{rad}^Z(r^{-3}).$$



For  $F_\alpha$  and  $G_\alpha$  we have pointwise bounds of the form

$$(2.19) \quad |F_\alpha| \lesssim \frac{1}{\langle r \rangle^3} (|\Omega^2 u_{<|\alpha|-6}| + |u_{<|\alpha|}|) + \frac{1}{\langle r \rangle^2} (|\nabla^2 u_{<|\alpha|}| + |\nabla u_{<|\alpha|}|) + |f_{\leq|\alpha|}|,$$

$$(2.20) \quad |G_\alpha| \lesssim \frac{1}{\langle r \rangle^3} (|\Omega^2 u_{\leq|\alpha|}| + |u_{\leq|\alpha|}|) + \frac{1}{\langle r \rangle^2} (|\nabla^2 u_{\leq|\alpha|}| + |\nabla u_{\leq|\alpha|}|) + |f_{\leq|\alpha|}|.$$

As a general principle, we will use the latter equation to improve the bounds on  $u_\alpha$  away from  $r = 0$  (precisely for  $r \gtrsim t$ ), and the former near  $r = 0$  (precisely for  $r \ll t$ ).

**2.2. The weak local energy decay.** The statement of the weak local energy decay property in Definition 1.3 includes the vector fields  $T$  but not  $S$  or  $\Omega$ . We remedy this in the following

**Lemma 2.6.** *Assume that the weak local energy decay property (1.9) holds. Then we also have*

$$(2.21) \quad \|u_{\leq m}\|_{LE^1} \lesssim \|\nabla u_{\leq m+1}(0)\|_{L^2} + \|f_{\leq m+1}\|_{LE^*}.$$

*Proof.* We use induction with respect to  $m$ . For  $m = 0$  the bounds (1.10) and (2.21) coincide. Consider now some  $m > 0$ , and  $\alpha$  a multiindex with  $|\alpha| = m$ . If  $Z^\alpha$  contains only  $T$  derivatives then the bound for  $u_\alpha$  follows directly from (1.10). Else we factor

$$Z^\alpha = T^i \Omega^j S^k.$$

Applying (1.10) to  $\Omega^j S^k u$  and using (2.16) we obtain

$$\begin{aligned} \|Z^\alpha u\|_{LE^1} &\lesssim \|\Omega^j S^k u\|_{LE^{i,1}} \\ &\lesssim \|\nabla \Omega^j S^k u(0)\|_{H^{i+1}} + \|\Omega^j S^k f\|_{LE^{i+1,*}} + \|Qu_{\leq 3j+9k-3}\|_{LE^{i+1,*}} \\ &\lesssim \|\nabla u_{\leq m+1}(0)\|_{L^2} + \|f_{\leq m+1}\|_{LE^*} + \|\langle r \rangle^{-2} \nabla u_{\leq m-1}\|_{LE^*} \\ &\quad + \|\langle r \rangle^{-3} u_{\leq m-1}\|_{LE^*} \\ &\lesssim \|\nabla u_{\leq m+1}(0)\|_{L^2} + \|f_{\leq m+1}\|_{LE^*} + \|u_{\leq m-1}\|_{LE^1} \end{aligned}$$

which concludes our induction.  $\square$

**2.3. The stationary local energy decay.** Our first aim here is to include the vector fields  $S$  and  $\Omega$  in the stationary local energy decay bounds. A second aim is to derive a variation of the same bounds with different weights.

**Lemma 2.7.** *Assume that the stationary local energy decay property (1.11) holds. Then for all  $0 \leq t_0 < t_1$  we also have*

$$(2.22) \quad \|u_{\leq m}\|_{LE^1[t_0, t_1]} \lesssim \sum_{i=0,1} \|\nabla u_{\leq m}(t_i)\|_{L^2} + \|f_{\leq m}\|_{LE^*} + \|\partial_t u_{\leq m}\|_{LE[t_0, t_1]},$$

respectively

$$(2.23) \quad \|\nabla u_{\leq m}\|_{L^2} \lesssim \sum_{i=0,1} \|\langle r \rangle^{\frac{1}{2}} \nabla u_{\leq m}(t_i)\|_{L^2} + \|\langle r \rangle f_{\leq m}\|_{L^2} + \|\partial_t u_{\leq m}\|_{L^2}.$$

*Proof.* The proof of (2.22) is identical to the proof of Lemma 2.6 and is omitted. We now prove (2.23). We begin with the case  $m = 0$ , where we apply the classical method due to Morawetz. Assume first that we are in Case A. Multiplying the equation  $Pu = f$  by  $(x\partial_x + 1)u$  and integrating by parts we obtain

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\mathbb{R}^3} Pu \cdot (x\partial_x + 1)u dx dt &= \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |\nabla u|^2 + O(\langle r \rangle^{-1})|\nabla u|^2 + O(\langle r \rangle^{-3})|u|^2 dx dt \\ &\quad + \int_{\mathbb{R}^3} O(\langle r \rangle)|\nabla u|^2 + O(\langle r \rangle^{-1})|u|^2 dx \Big|_{t=t_0}^{t=t_1}. \end{aligned}$$

Using Cauchy-Schwarz on the left and estimating the  $|u|^2$  terms by  $|\nabla u|^2$  terms via Hardy inequalities we are left with

$$LHS(2.23)(m=0) \lesssim RHS(2.23)(m=0) + \|\nabla u\|_{L^2(r \leq R)} + \|u\|_{L^2(r \leq R)}$$

for some fixed large  $R$ . Here the extraneous terms on the right are only measured for small  $r$ , as the large  $r$  contribution can be absorbed on the left. However, to bound them for small  $r$  we have at our disposal the bound (2.22), whose right hand side is smaller than the right hand side of (2.23). The same outcome is reached in Case B by inserting a cutoff function selecting the region  $\{r \gg 1\}$  in the above computation.

To prove (2.23) we can use a simpler direct argument since there is no loss of derivatives on one hand and since we already have the bound (2.22) to use to estimate  $u_{\leq m}$  inside a compact set. Precisely, for  $|\alpha| = m$  we have

$$Pu_\alpha = f_{\leq m} + O(\langle r \rangle^{-2})\nabla u_{\leq m} + O(\langle r \rangle^{-3})u_{\leq m}.$$

Then we apply the  $m = 0$  case of (2.23) for  $u_\alpha$  and sum over  $|\alpha| \leq m$ . We obtain

$$LHS(2.23) \lesssim RHS(2.23) + \|\nabla u_{\leq m}\|_{L^2(r \leq R)} + \|u_{\leq m}\|_{L^2(r \leq R)}$$

and the last terms on the right are estimated by (2.22).  $\square$

### 3. THE POINTWISE DECAY

The strategy of the proof of our pointwise decay estimates is to iteratively improve the estimates via a two step approach. The two steps are as follows:

(i) Use the properties of the fundamental solution for the constant coefficient d'Alembertian  $\square$  via the equation (2.18). This yields improved bounds for  $r \gg 1$ , but no improvement at all for  $r \sim 1$ .

(ii) Use the stationary local energy decay estimates for the operator  $P$  in the region  $r \ll t$ . This allows us to obtain improved bounds for small  $r$ . The transition from  $L^2$  to pointwise bounds is done in a standard manner via Sobolev type estimates.

**3.1. The cone decomposition and Sobolev embeddings.** For the forward cone  $C = \{r \leq t\}$  we consider a dyadic decomposition in time into sets

$$C_T = \{T \leq t \leq 2T, \quad r \leq t\}.$$

For each  $C_T$  we need a further double dyadic decomposition of it with respect to either the size of  $t - r$  or the size of  $r$ , depending on whether we are close or far from the cone,

$$C_T = \bigcup_{1 \leq R \leq T/4} C_T^R \cup \bigcup_{1 \leq U < T/4} C_T^U$$

where for  $R, U > 1$  we set

$$C_T^R = C_T \cap \{R < r < 2R\}, \quad C_T^U = C_T \cap \{U < t - r < 2U\}$$

while for  $R = 1$  and  $U = 1$  we have

$$C_T^{R=1} = C_T \cap \{0 < r < 2\}, \quad C_T^{U=1} = C_T \cap \{0 < t - r < 2\}$$

with the obvious change for  $C_T^1$  in Case B. By  $\tilde{C}_T^R$  and  $\tilde{C}_T^U$  we denote enlargements of these sets in both space and time on their respective scales. We also define

$$C_T^{<T/2} = \bigcup_{R < T/4} C_T^R.$$

The sets  $C_T^R$  and  $C_T^U$  represent the setting in which we apply Sobolev embeddings, which allow us to obtain pointwise bounds from  $L^2$  bounds. Precisely, we have

**Lemma 3.8.** *For all  $T \geq 1$  and  $1 \leq R, U \leq T/4$  we have*  
(3.24)

$$\|w\|_{L^\infty(C_T^R)} \lesssim \frac{1}{T^{\frac{1}{2}} R^{\frac{3}{2}}} \sum_{i+j \leq 2} \|S^i \Omega^j w\|_{L^2(\tilde{C}_T^R)} + \frac{1}{T^{\frac{1}{2}} R^{\frac{1}{2}}} \sum_{i+j \leq 2} \|S^i \Omega^j \nabla w\|_{L^2(\tilde{C}_T^R)},$$

respectively

$$(3.25) \quad \|w\|_{L^\infty(C_T^U)} \lesssim \frac{1}{T^{\frac{3}{2}} U^{\frac{1}{2}}} \sum_{i+j \leq 2} \|S^i \Omega^j w\|_{L^2(\tilde{C}_T^U)} + \frac{U^{\frac{1}{2}}}{T^{\frac{3}{2}}} \sum_{i+j \leq 2} \|S^i \Omega^j \nabla w\|_{L^2(\tilde{C}_T^U)}.$$

*Proof.* In exponential coordinates  $(s, \rho, \omega)$  with  $t = e^s$  and  $r = e^{s+\rho}$  the bound (3.24) is nothing but the usual Sobolev embedding applied uniformly in regions of size one. The same applies for (3.25) in exponential coordinates  $(s, \rho, \omega)$  with  $t = e^s$  and  $t - r = e^{s+\rho}$ .  $\square$

Expressed in terms of the local energy norm, the estimate (3.24) yields

**Corollary 3.9.** *We have*

$$(3.26) \quad \|w\|_{L^\infty(C_T^{<T/2})} \lesssim T^{-\frac{1}{2}} \sum_{i+j \leq 2} \|S^i \Omega^j w\|_{LE^1(\tilde{C}_T^{<T/2})}.$$

**3.2. The one dimensional reduction.** A main method to obtain pointwise estimates for  $u_\alpha$  is by using the positivity of the fundamental solution to the wave equation in  $3+1$  dimensions and the standard one dimensional reduction.

For solutions to (2.18) with vanishing initial data, we may apply time translation invariance and assume without loss of generality that  $G_\alpha$  is supported in  $C = \{r \leq t\}$ . Then define

$$(3.27) \quad H_\alpha(t, r) = \sum_0^2 \|\Omega^i G_\alpha(t, r\omega)\|_{L^2(\mathbb{S}^2)},$$

for  $G_\alpha$  as in (2.18). By the Sobolev embeddings on the sphere we know that  $|G_\alpha| \leq H_\alpha$ . Let  $v_\alpha$  be the radial solution to

$$(3.28) \quad \square v_\alpha = H_\alpha, \quad v_\alpha(0) = \partial_t v_\alpha(0) = 0.$$

Then we can compare

$$|u_\alpha| \leq v_\alpha.$$

We can rewrite the radial three dimensional equation (3.28) as a one dimensional problem

$$(\partial_t^2 - \partial_r^2)(rv_\alpha) = rH_\alpha$$

which has the solution

$$(rv_\alpha)(t, r) = \frac{1}{2} \int_0^t \int_{|r-t+s|}^{r+t-s} \rho H_\alpha(s, \rho) d\rho ds.$$

Assuming that  $H_\alpha$  is supported in the forward cone  $t \geq r$ , this is rewritten as

$$(3.29) \quad rv_\alpha(t, r) = \frac{1}{2} \int_{D_{tr}} \rho H_\alpha(s, \rho) ds d\rho$$

where  $D_{tr}$  is the rectangle

$$D_{tr} = \{0 \leq s - \rho \leq t - r, \quad t - r \leq s + \rho \leq t + r\}.$$

In order to handle the contribution from the initial data in both cases and the fact that (2.18) only holds outside of the cylinder  $\mathbb{R}^+ \times B(0, R_0)$  in Case B, we modify the above argument. The one dimensional reduction is only used to improve the bounds on  $u_\alpha$  for large  $r$ . Hence we can truncate the functions  $u_\alpha$  outside a large ball using a cutoff function  $\chi_{out}$  which is identically one for large  $r$ . Then we can redefine  $G_\alpha$  as

$$G_\alpha = \square(\chi_{out} u_\alpha).$$

With this choice for  $G_\alpha$  the bound (2.20) still holds. Moreover by truncating outside of a sufficiently large ball,  $\chi_{out} u_\alpha$  has vanishing initial data. We can use the one dimensional reduction to obtain bounds for  $u_\alpha$  for large  $r$ , while for small  $r$ , we shall rely on the Sobolev-type estimates of Lemma 3.8.

**3.3. An initial decay bound.** Here we combine the above one dimensional reduction with the local energy bounds in order to obtain an initial pointwise decay estimate for the functions  $u_\alpha$ . This has the form

**Lemma 3.10.** *The following estimate holds:*

$$(3.30) \quad |u_\alpha| \lesssim \frac{\log\langle t - r \rangle}{\langle r \rangle \langle t - r \rangle^{\frac{1}{2}}} (\|u_{\leq \alpha+n}\|_{LE^1} + \|\langle r \rangle f_{\leq \alpha+n}\|_{LE^*})$$

Here and later in this section  $n$  represents a large constant which does not depend on  $\alpha$  but may increase from one subsection to the next. For the above lemma we can take  $n = 25$  for instance, but later on it becomes tedious and not particularly illuminating to keep track of the exact value of  $n$ . We shall do so similarly for the enlargements of the cones  $\tilde{C}_T^R, \tilde{C}_T^U$ . We shall not track, though we note that only a finite number will ever be required, each subsequent enlargement which is needed and shall allow the enlargement to change from line to line while maintaining the same notation.

*Proof.* We assume that  $r \gg 1$ . For  $r \lesssim 1$  we instead use directly the Sobolev type embedding (3.26).

For large  $r$ , we bound  $u_\alpha$  by  $v_\alpha$  and the function  $H_\alpha$ , using (2.20), by

$$\|\langle r \rangle^2 H_\alpha\|_{LE} + \|\langle r \rangle^2 S H_\alpha\|_{LE} \lesssim \|u_{\leq \alpha+25}\|_{LE^1} + \|\langle r \rangle f_{\leq \alpha+25}\|_{LE^*}.$$

Hence it remains to show that the solution  $v_\alpha$  to (3.28) satisfies

$$(3.31) \quad |v_\alpha| \lesssim \frac{\log\langle t - r \rangle}{\langle r \rangle \langle t - r \rangle^{\frac{1}{2}}} (\|\langle r \rangle^2 H_\alpha\|_{LE} + \|\langle r \rangle^2 S H_\alpha\|_{LE}).$$

We now prove (3.31). The index  $\alpha$  plays no role in it so we drop it. One can then estimate  $|rv|$  as

$$(3.32) \quad |rv(t, r)| \lesssim \int_{D_{tr}} \rho H(s, \rho) ds d\rho.$$

We assume that  $r \sim t$ , as there is no further gain for smaller  $r$  in estimating the integral on the right.

We partition the set  $D_{tr}$  into a double dyadic manner as

$$D_{tr} = \bigcup_{R \leq t} D_{tr}^R, \quad D_{tr}^R = D_{tr} \cap \{R < r < 2R\}$$

and estimate the corresponding parts of the above integral. We consider two cases:

(i)  $R < (t - r)/8$ . Then we need to use the information about  $Su$ . For any  $(s, \rho) \in C$ , let  $\gamma_{s, \rho}(\tau)$  be the integral curve corresponding to the vector field  $S$ , parametrized by time, satisfying  $\gamma_{s, \rho}(0) = (s, \rho)$ . The fundamental theorem of calculus combined with the Cauchy-Schwarz inequality gives

$$|H(\gamma_{s, \rho}(0))|^2 \leq \frac{1}{s} \int_0^s |H(\gamma_{s, \rho}(\tau))|^2 d\tau + \frac{1}{s} \int_0^s |(SH)(\gamma_{s, \rho}(\tau))|^2 d\tau.$$

We apply this for  $(s, \rho) \in D_{tr}^R$  and integrate. In the region  $D_{tr}^R$  we have  $\rho \sim R$  and  $|s - (t - r)| \lesssim R$ ; therefore we obtain

$$\int_{D_{tr}^R} |H|^2 ds d\rho \lesssim \frac{R}{t - r} \int_{B_R} |H|^2 + |SH|^2 ds d\rho$$

where

$$B_R = \{(s, \rho) : R/8 < \rho < 8R\}.$$

Hence by Cauchy-Schwarz we conclude

$$\begin{aligned} \int_{D_{tr}^R} \rho H ds d\rho &\lesssim \frac{R^{\frac{5}{2}}}{\langle t - r \rangle^{\frac{1}{2}}} (\|H\|_{L^2(B_R)} + \|SH\|_{L^2(B_R)}) \\ &\lesssim \frac{1}{\langle t - r \rangle^{\frac{1}{2}}} (\|\langle r \rangle^2 H\|_{LE} + \|\langle r \rangle^2 SH\|_{LE}). \end{aligned}$$

The logarithmic factor in (3.31) arises in the dyadic  $R$  summation. We note that in the  $L^2(B_R)$  norm above  $H$  is viewed as a two dimensional function, whereas the  $LE$  norm applies to  $H$  as a radial function in  $3 + 1$  dimensions.

(ii)  $(t - r)/8 < R < t$ . Then we neglect  $SH$  and simply use Cauchy-Schwarz,

$$\int_{D_{tr}^R} \rho H d\rho ds \lesssim R^{\frac{3}{2}} (t - r)^{\frac{1}{2}} \|H\|_{L^2(B_R)} \lesssim R^{-1} (t - r)^{\frac{1}{2}} \|\langle r \rangle^2 H\|_{LE}.$$

The dyadic  $R$  summation is again straightforward.  $\square$

**3.4. Improved  $L^2$  gradient bounds.** The bounds obtained in the previous step for  $u_\alpha$  apply as well to  $\nabla u_\alpha$ . However,  $u_\alpha$  and  $\nabla u_\alpha$  do not play symmetrical roles in the expressions for  $F_\alpha$  and  $G_\alpha$ . In particular, the weights that come with  $\nabla u_\alpha$  are worse than the ones that come with  $u_\alpha$ . Hence, when we seek to reiterate and improve the initial pointwise bound (3.30) it pays to have better bounds for  $\nabla u_\alpha$ . This is the aim of this step in the proof. Our dyadic  $L^2$  gradient bound is contained in the next lemma:

**Lemma 3.11.** *For  $1 \ll U, R \leq T/4$  we have*

$$(3.33) \quad \|\nabla w\|_{L^2(C_T^R)} \lesssim R^{-1}\|w\|_{L^2(\tilde{C}_T^R)} + T^{-1}\|Sw\|_{L^2(\tilde{C}_T^R)} + R\|Pw\|_{L^2(\tilde{C}_T^R)}$$

*respectively*

$$(3.34) \quad \|\nabla w\|_{L^2(C_T^U)} \lesssim U^{-1}(\|w\|_{L^2(\tilde{C}_T^U)} + \|Sw\|_{L^2(\tilde{C}_T^U)}) + T\|Pw\|_{L^2(\tilde{C}_T^U)}$$

Applied to  $u_\alpha$  this gives

**Corollary 3.12.** *For  $1 \ll U, R \leq T/4$  we have*

$$(3.35) \quad \|\nabla u_\alpha\|_{L^2(C_T^R)} \lesssim R^{-1}\|u_{\leq \alpha+n}\|_{L^2(\tilde{C}_T^R)} + R\|f_{\leq \alpha}\|_{L^2(\tilde{C}_T^R)}$$

*respectively*

$$(3.36) \quad \|\nabla u_\alpha\|_{L^2(C_T^U)} \lesssim U^{-1}\|u_{\leq \alpha+n}\|_{L^2(\tilde{C}_T^U)} + T\|f_{\leq \alpha}\|_{L^2(\tilde{C}_T^U)}$$

Applied to  $\nabla u_\alpha$  we also obtain

**Corollary 3.13.** *For  $1 \ll U, R \leq T/4$  we have*

$$(3.37) \quad \|\nabla^2 u_\alpha\|_{L^2(C_T^R)} \lesssim R^{-1}\|\nabla u_{\leq \alpha+n}\|_{L^2(\tilde{C}_T^R)} + R\|\nabla f_{\leq \alpha}\|_{L^2(\tilde{C}_T^R)}$$

*respectively*

$$(3.38) \quad \|\nabla^2 u_\alpha\|_{L^2(C_T^U)} \lesssim U^{-1}\|\nabla u_{\leq \alpha+n}\|_{L^2(\tilde{C}_T^U)} + T\|\nabla f_{\leq \alpha}\|_{L^2(\tilde{C}_T^U)}$$

*Proof of Lemma 3.11.* To keep the ideas clear we first prove the lemma with  $P$  replaced by  $\square$ . We consider a cutoff function  $\beta$  supported in  $\tilde{C}_T^R$  which equals 1 on  $C_T^R$ . Integration by parts gives

$$(3.39) \quad \int \beta(|\nabla_x w|^2 - |\partial_t w|^2) dx dt = \int \square w \cdot \beta w dx dt - \frac{1}{2} \int (\square \beta) w^2 dx dt.$$

To estimate  $\nabla w$  we use the pointwise inequality

$$|\nabla w|^2 \leq M \frac{1}{(t-r)^2} |Sw|^2 + \frac{t}{t-r} (|\nabla_x w|^2 - |\partial_t w|^2)$$

which is valid inside the cone  $C$  for a fixed large  $M$ . Hence

$$\int \beta |\nabla w|^2 dx dt \lesssim \int \frac{1}{(t-r)^2} \beta |Sw|^2 + \frac{t}{t-r} |\square \beta| w^2 + \frac{t}{t-r} \beta |\square w| |w| dx dt$$

where all weights have a fixed size in the support of  $\beta$ . The function  $\beta$  can be further chosen so that  $|\square \beta| \lesssim r^{-2}$ . Then the conclusion of the lemma follows by applying Cauchy-Schwarz to the last term. The argument for  $C_T^U$  is similar, with the only difference that now we have  $|\square \beta| \lesssim t^{-1}(t-r)^{-1}$ .

Now consider the above proof but with  $\square$  replaced by  $P$ . Then, given the form of  $P$  in (1.2), (1.3), the relation (3.39) is modified as follows:

$$\begin{aligned} \int \beta (|\nabla_x w|^2 - |\partial_t w|^2) dx dt &= \int Pw \cdot \beta w dx dt - \frac{1}{2} \int ((P+V)\beta) w^2 dx dt \\ &\quad + \int O(r^{-1})\beta |\nabla w|^2 dx dt. \end{aligned}$$

The bound for  $(P+V)\beta$  is similar to the bound for  $\square \beta$ , and one can easily see that the last error term is harmless. The proof of the lemma is concluded.  $\square$

**3.5. Improved  $L^2$  bounds for small  $r$ .** A very unsatisfactory feature of our first pointwise bound (3.30) is the  $r^{-1}$  factor which is quite bad for small  $r$ . Here we devise a mechanism which allows us to replace this factor by  $t^{-1}$ . Our main bound is an  $L^2$  local energy bound, derived using the stationary local energy decay assumption. We have

**Proposition 3.14.** *Assume that the problem (1.1) satisfies stationary local energy decay bounds (1.11). Then the following estimates hold:*

$$(3.40) \quad \|u_{\leq m}\|_{LE^1(C_T^{\leq T/2})} \lesssim T^{-1} \|\langle r \rangle u_{\leq m+n}\|_{LE^1(C_T^{\leq T/2})} + \|f_{\leq m+n}\|_{LE^*(C_T^{\leq T/2})}.$$

*Proof.* We first observe that we can harmlessly truncate  $u$  to  $C_T^{\leq T/2}$ . We will make this assumption throughout. Applying the stationary local energy decay estimate (2.22) for  $u_{\leq m}$  we obtain

$$\begin{aligned} \|u_{\leq m}\|_{LE^1(C_T)} &\lesssim \|\nabla u_{\leq m}(T)\|_{L^2} + \|\nabla u_{\leq m}(2T)\|_{L^2} \\ &\quad + \|f_{\leq m}\|_{LE^*(C_T)} + \|\partial_t u_{\leq m}\|_{LE(C_T)}. \end{aligned}$$

For the last term on the right, due to the support of  $u$ , we have

$$\begin{aligned} \|\partial_t u_{\leq m}\|_{LE(C_T)} &\lesssim \frac{1}{T} (\|Su_{\leq m}\|_{LE(C_T)} + \|r\partial_r u_{\leq m}\|_{LE(C_T)}) \\ &\lesssim T^{-\frac{1}{2}} (\|\langle r \rangle^{-1} Su_{\leq m}\|_{L^2} + \|\nabla u_{\leq m}\|_{L^2}) \\ &\lesssim T^{-\frac{1}{2}} (\|\nabla Su_{\leq m}\|_{L^2} + \|\nabla u_{\leq m}\|_{L^2}) \end{aligned}$$

where a Hardy inequality was used in the last step. Next we consider the time boundary terms. By the fundamental theorem of calculus and the Cauchy-Schwarz inequality, for any  $s > 0$  we have

$$(3.41) \quad |\nabla u(T, x)|^2 \lesssim \frac{1}{s} \int_0^s |(\nabla u)(\gamma_{T,x}(\tau))|^2 d\tau + \frac{s}{T^2} \int_0^s |(S\nabla u)(\gamma_{T,x}(\tau))|^2 d\tau.$$

This holds uniformly with respect to  $s$ , so we have the freedom to choose  $0 < s \leq T$  favourably depending on  $x$ . Suppose we take  $s = T$ . Integrating the above estimate over  $|x| \leq T$  for this choice of  $s$  and applying a change of coordinates yields

$$\int_{C_T \cap \{t=T\}} |\nabla u_{\leq m}(T)|^2 dx \lesssim T^{-1} \int_{C_T} |\nabla u_{\leq m}|^2 + |S\nabla u_{\leq m}|^2 dx dt.$$

A similar bound holds for  $\nabla u_{\leq m}(2T)$ . Combining the last three estimates we obtain

$$(3.42) \quad \|u_{\leq m}\|_{LE^1(C_T)} \lesssim \|f_{\leq m+n}\|_{LE^*(C_T)} + T^{-\frac{1}{2}} \|\nabla u_{\leq m+n}\|_{L^2(C_T)}$$

which brings us half of the way to the proof of (3.40).

We now consider the second expression on the right above, and we estimate it via the same argument as before, but using (2.23) instead of (2.22). Indeed, (2.23) yields

$$\begin{aligned} \|\nabla u_{\leq m}\|_{L^2(C_T)} &\lesssim \|\langle r \rangle f_{\leq m}\|_{L^2(C_T)} + \|\langle r \rangle^{\frac{1}{2}} \nabla u_{\leq m}(T)\|_{L^2} + \|\langle r \rangle^{\frac{1}{2}} \nabla u_{\leq m}(2T)\|_{L^2} \\ &\quad + \|\partial_t u_{\leq m}\|_{L^2(C_T)}. \end{aligned}$$

The last term is controlled as above by

$$\begin{aligned} \|\partial_t u_{\leq m}\|_{L^2(C_T)} &\lesssim T^{-1} (\|Su_{\leq m}\|_{L^2(C_T)} + \|r\nabla u_{\leq m}\|_{L^2(C_T)}) \\ &\lesssim T^{-\frac{1}{2}} (\|\langle r \rangle Su_{\leq m}\|_{LE^1(C_T)} + \|\langle r \rangle u_{\leq m}\|_{LE^1(C_T)}). \end{aligned}$$

The initial and final terms are estimated by a more careful use of (3.41). Namely, we integrate (3.41) with  $s(x) = \langle r \rangle^{\frac{1}{2}} T^{\frac{1}{2}}$ . This yields

$$\int \langle r \rangle |\nabla u_{\leq m}(T)|^2 dx \lesssim T^{-\frac{1}{2}} \int_{C_T} \langle r \rangle^{\frac{1}{2}} |\nabla u_{\leq m}|^2 + T^{-\frac{3}{2}} \int_{C_T} \langle r \rangle^{\frac{3}{2}} |S \nabla u_{\leq m}|^2 dx dt$$

which implies that

$$\|\langle r \rangle^{\frac{1}{2}} u_{\leq m}(T)\|_{L^2} \lesssim T^{-\frac{1}{4}} \|\langle r \rangle^{\frac{1}{4}} \nabla u_{\leq m}\|_{L^2(C_T)} + T^{-\frac{1}{2}} \|\langle r \rangle S u_{\leq m}\|_{L^1(C_T)}.$$

Combining the last three bounds leads to

$$\begin{aligned} \|\nabla u_{\leq m}\|_{L^2(C_T)} &\lesssim \|\langle r \rangle f_{\leq m+n}\|_{L^2(C_T)} + T^{-\frac{1}{2}} \|\langle r \rangle u_{\leq m+n}\|_{L^1(C_T)} \\ &\quad + T^{-\frac{1}{4}} \|\langle r \rangle^{\frac{1}{4}} \nabla u_{\leq m}\|_{L^2(C_T)}. \end{aligned}$$

We can discard the last term on the right by absorbing it into the left hand side for  $r \ll T$  and into the second right hand side term for  $r \sim T$ . Thus we obtain

$$(3.43) \quad \|\nabla u_{\leq m}\|_{L^2(C_T)} \lesssim \|\langle r \rangle f_{\leq m+n}\|_{L^2(C_T)} + T^{-\frac{1}{2}} \|\langle r \rangle u_{\leq m+n}\|_{L^1(C_T)}.$$

Finally, the bound (3.40) is obtained by combining (3.42) and (3.43).  $\square$

**3.6. Improved pointwise bounds for  $u_\alpha$  for small  $r$ .** Here we use Sobolev type inequalities to make the transition from  $L^2$  local energy bounds to pointwise bounds for small  $r$ . Our main estimate has the form

**Proposition 3.15.** *We have*

$$(3.44) \quad \begin{aligned} \|u_{\leq m}\|_{L^\infty(C_T^{\leq T/2})} + \|\langle r \rangle \nabla u_{\leq m}\|_{L^\infty(C_T^{\leq T/2})} &\lesssim T^{-\frac{3}{2}} \|u_{\leq m+n}\|_{LE(\tilde{C}_T^{\leq T/2})} \\ &\quad + T^{-\frac{1}{2}} \left( \|f_{\leq m+n}\|_{LE^*(\tilde{C}_T^{\leq T/2})} + \|\langle r \rangle^2 \nabla f_{\leq m+n}\|_{LE(\tilde{C}_T^{\leq T/2})} \right). \end{aligned}$$

*Proof.* In view of the Sobolev bound (3.26) in Corollary 3.9, the estimate (3.44) would follow from

$$(3.45) \quad \begin{aligned} \|u_{\leq m}\|_{LE^1(C_T^{\leq T/2})} + \|\langle r \rangle \nabla u_{\leq m}\|_{LE^1(C_T^{\leq T/2})} &\lesssim T^{-1} \|u_{\leq m+n}\|_{LE(\tilde{C}_T^{\leq T/2})} \\ &\quad + \|f_{\leq m+n}\|_{LE^*(\tilde{C}_T^{\leq T/2})} + \|\langle r \rangle^2 \nabla f_{\leq m+n}\|_{LE(\tilde{C}_T^{\leq T/2})}. \end{aligned}$$

To prove this we begin with the local energy bound (3.40) and add to it corresponding bounds for the second derivatives of  $u_{\leq m}$ . For that we use (3.37); applied uniformly in dyadic regions  $C_T^R$  with  $R \lesssim T$  it yields

$$\|\langle r \rangle \nabla^2 u_{\leq m}\|_{LE(C_T^{\leq T/2})} \lesssim \|u_{\leq m+n}\|_{LE^1(C_T^{\leq T/2})} + \|\langle r \rangle^2 \nabla f_{\leq m+n}\|_{LE(C_T^{\leq T/2})}.$$

Combined with (3.40), this leads to

$$(3.46) \quad \begin{aligned} LHS(3.45) &\lesssim T^{-1} \|\langle r \rangle u_{\leq m+n}\|_{LE^1(\tilde{C}_T^{\leq T/2})} + \|f_{\leq m+n}\|_{LE^*(C_T^{\leq T/2})} \\ &\quad + \|\langle r \rangle^2 \nabla f_{\leq m+n}\|_{LE(C_T^{\leq T/2})}. \end{aligned}$$

It remains to make the transition from (3.46) to (3.45). This is done via (3.35) applied uniformly in dyadic regions  $C_T^R$  with  $R \lesssim T$ , which yields

$$\|\langle r \rangle u_{\leq m}\|_{LE^1(\tilde{C}_T^{\leq T/2})} \lesssim \|u_{\leq m+n}\|_{LE(\tilde{C}_T^{\leq T/2})} + \|\langle r \rangle^2 f_{\leq m+n}\|_{LE(\tilde{C}_T^{\leq T/2})}.$$

Hence (3.45) follows.  $\square$



**3.7. Improved pointwise estimates for  $\nabla u_\alpha$  near the cone.** Here we convert the improvement in the  $L^2$  bounds for  $\nabla u_{\leq m}$  given by Lemma 3.11 near the cone into a similar  $L^\infty$  bound. This shows that the pointwise bounds for  $\nabla u_{\leq m}$  are better than those for  $u_{\leq m}$  by a factor of  $\langle t-r \rangle^{-1}$ .

**Proposition 3.16.** *We have*

$$(3.47) \quad \begin{aligned} U \|\nabla u_{\leq m}\|_{L^\infty(C_T^U)} &\lesssim \|u_{\leq m+n}\|_{L^\infty(\tilde{C}_T^U)} + T^{-\frac{1}{2}} U^{\frac{1}{2}} \|f_{\leq m+n}\|_{L^2(\tilde{C}_T^U)} \\ &\quad + T^{-\frac{1}{2}} U^{\frac{3}{2}} \|\nabla f_{\leq m+n}\|_{L^2(\tilde{C}_T^U)}. \end{aligned}$$

*Proof.* The proof is similar to the proof of Proposition 3.15. In view of the Sobolev inequality (3.25), the bound (3.47) would follow from

$$\begin{aligned} U \|\nabla u_{\leq m}\|_{L^2(C_T^U)} + U^2 \|\nabla^2 u_{\leq m}\|_{L^2(C_T^U)} &\lesssim \|u_{\leq m+n}\|_{L^2(C_T^U)} + UT \|f_{\leq m+n}\|_{L^2(C_T^U)} \\ &\quad + U^2 T \|\nabla f_{\leq m+n}\|_{L^2(C_T^U)}. \end{aligned}$$

To prove the bound on  $\nabla u_{\leq m}$  we use (3.36), which shows that

$$U \|\nabla u_{\leq m}\|_{L^2(C_T^U)} \lesssim \|u_{\leq m+n}\|_{L^2(\tilde{C}_T^U)} + UT \|f_{\leq m+n}\|_{L^2(\tilde{C}_T^U)}.$$

To prove the bound on  $\nabla^2 u_{\leq m}$  we use (3.38) to obtain

$$U^2 \|\nabla^2 u_{\leq m}\|_{L^2(C_T^U)} \lesssim U \|\nabla u_{\leq m+n}\|_{L^2(\tilde{C}_T^U)} + U^2 T \|\nabla f_{\leq m+n}\|_{L^2(\tilde{C}_T^U)}.$$

□

**3.8. Gradient bounds associated to the first decay bound.** Here we start with the bound (3.30) for  $u_{\leq m}$  and improve it for small  $r$ , as well as complement it with gradient bounds.

**Lemma 3.17.** *The following pointwise estimates hold:*

$$(3.48) \quad |u_{\leq m}| \lesssim C_1 \frac{\log\langle t-r \rangle}{t\langle t-r \rangle^{\frac{1}{2}}}, \quad |\nabla u_{\leq m}| \lesssim C_1 \frac{\log\langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^{\frac{3}{2}}}$$

where

$$C_1 = \|u_{\leq m+n}\|_{LE^1} + \sup_{R,U} T^{\frac{1}{2}} R^{\frac{1}{2}} U^{\frac{1}{2}} \|f_{\leq m+n}\|_{L^2(C_T^{R,U})} + T^{-\frac{1}{2}} R^{\frac{3}{2}} U^{\frac{3}{2}} \|\nabla f_{\leq m+n}\|_{L^2(C_T^{R,U})}.$$

Here for brevity  $C_T^{R,U}$  stands for either  $C_T^R$  or  $C_T^U$ , with the natural convention that  $R \sim T$  in  $C_T^U$  and  $U \sim T$  in  $C_T^R$ . The proof is a direct application of Propositions 3.15, 3.16, using (3.30) as a starting point.

**3.9. The second decay bound.**

**Lemma 3.18.** *The following decay estimate holds:*

$$(3.49) \quad |u_{\leq m}| \lesssim C_2 \frac{\log\langle t-r \rangle}{t\langle t-r \rangle}, \quad |\nabla u_{\leq m}| \lesssim C_2 \frac{\log\langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^2}$$

where

$$C_2 = \|u_{\leq m+n}\|_{LE^1} + \sup_{R,U} TR^{\frac{1}{2}} U^{\frac{1}{2}} \|f_{\leq m+n}\|_{L^2(C_T^{R,U})} + R^{\frac{3}{2}} U^{\frac{3}{2}} \|\nabla f_{\leq m+n}\|_{L^2(C_T^{R,U})}.$$

*Proof.* By the Sobolev embeddings of Lemma 3.8 we have

$$|f_{\leq m}| \lesssim \frac{1}{t\langle r \rangle^2 \langle t-r \rangle} C_2.$$

Also by (3.48) we have

$$|u_{\leq m+6}| \lesssim \frac{\log \langle t-r \rangle}{t \langle t-r \rangle^{\frac{1}{2}}} C_2, \quad |\nabla u_{\leq m+6}| \lesssim \frac{\log \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^{\frac{3}{2}}} C_2.$$

Hence for the functions  $G_{\leq m}$  we obtain

$$|G_{\leq m}| \lesssim \frac{1}{\langle r \rangle^3 \langle t-r \rangle} C_2.$$

Computing via the one dimensional reduction, this leads to a bound for  $u_{\leq m}$  of the form

$$|u_{\leq m}| \lesssim C_2 \frac{\log \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle}$$

which is comparable to (3.49) near the cone, but it is weaker for  $r \ll t$ . Then the small  $r$  bound for  $u_{\leq m}$  and the bound for  $\nabla u_{\leq m}$  are obtained from Propositions 3.15, 3.16 (we need to increase  $n$  appropriately at this stage).  $\square$

### 3.10. The third decay bound.

**Lemma 3.19.** *The following decay estimate holds:*

$$(3.50) \quad |u_{\leq m}| \lesssim C_3 \frac{\log^3 \langle t-r \rangle}{t \langle t-r \rangle^2}, \quad |\nabla u_{\leq m}| \lesssim C_3 \frac{\log^3 \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^3}$$

where

$$C_3 = \|u_{\leq m+n}\|_{LE^1} + \sup T^2 R^{\frac{1}{2}} U^{\frac{1}{2}} \|f_{\leq m+n}\|_{L^2(C_T^{R,U})} + T R^{\frac{3}{2}} U^{\frac{3}{2}} \|\nabla f_{\leq m+n}\|_{L^2(C_T^{R,U})}.$$

*Proof.* As before, the main step in the proof is to obtain a pointwise bound for  $u_{\leq m}$  which coincides with (3.50) near the cone,

$$(3.51) \quad |u_{\leq m}| \lesssim \frac{\log^3 \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^2} C_3.$$

Once this is done, the full estimate (3.50) follows easily by a direct application of Propositions 3.16, 3.15. However, at this stage there is a new twist, namely that the one dimensional reduction no longer suffices for the proof of (3.51).

The pointwise bound for  $f_{\leq m}$  has the form

$$|f_{\leq m}| \lesssim \frac{1}{t^2 \langle r \rangle^2 \langle t-r \rangle} C_3.$$

Also by (3.49) we have

$$|u_{\leq m+6}| \lesssim \frac{\log \langle t-r \rangle}{t \langle t-r \rangle} C_3, \quad |\nabla u_{\leq m+6}| \lesssim \frac{\log \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^2} C_3.$$

We use the wave equation for  $u_{\leq m}$  given by (2.18), and rewrite  $G_{\leq m}$  in the form

$$G_{\leq m} = f_{\leq m} + a u_{\leq m+6} + \partial_t (b u_{\leq m+6}), \quad a \in S^Z(r^{-3}), \quad b \in S^Z(r^{-2}).$$

Here we can confine ourselves to  $\partial_t$  derivatives in the last term because for any  $S$  and  $\Omega$  component we gain a factor of  $r^{-1}$  and include it in the second term. We split  $G_{\leq m}$  into two parts,

$$G_{\leq m} = G_{\leq m}^1 + G_{\leq m}^2$$

with

$$G_{\leq m}^1 = f_{\leq m} + au_{\leq m+6} + \partial_t((1 - \chi(t, r))bu_{< m+6}), \quad G_{\leq m}^2 = \partial_t(\chi(t, r)bu_{< m+6})$$

Here  $\chi$  is a smooth cutoff selecting the region  $t - r \ll t$ .

The function  $G_{\leq m}^1$  contains the part of  $G_{\leq m}$  which has good pointwise bounds,

$$|G_{\leq m}^1| \lesssim \frac{\log \langle t - r \rangle}{t \langle r \rangle^3 \langle t - r \rangle} C_3.$$

Computing via the one dimensional reduction, this gives the pointwise bound (3.51) for the corresponding part  $u_{\leq m}^1$  of  $u_{\leq m}$ .

Next we prove the same bound for the output  $u_{\leq m}^2$  of  $G_{\leq m}^2$ . Taking absolute values and applying the one dimensional reduction does not work, as it misses a cancellation due to the presence of the derivatives. Instead we do a more precise computation

**Lemma 3.20.** *Consider a smooth function  $f$  supported in  $\{\frac{t}{2} \leq r \leq t\}$  so that*

$$(3.52) \quad |f| + |Sf| + |\Omega f| + \langle t - r \rangle |\partial_r f| \lesssim \frac{1}{t^3 \langle t - r \rangle \log^2 \langle t - r \rangle}.$$

*Then the forward solution  $u$  to*

$$\square u = \partial_t f$$

*satisfies the bound*

$$(3.53) \quad |u| \lesssim \frac{1}{t \langle t - r \rangle^2}.$$

We note that if (3.52) is replaced by

$$(3.54) \quad |f| + |\nabla f| + |Sf| + |\Omega f| + \langle t - r \rangle |\partial_r f| \lesssim \frac{\log \langle t - r \rangle}{t^3 \langle t - r \rangle}$$

then (3.53) is trivially replaced by

$$(3.55) \quad |u| \lesssim \frac{\log^3 \langle t - r \rangle}{t \langle t - r \rangle^2}$$

which suffices to conclude the proof of Lemma 3.19.

*Proof.* The function  $u$  is expressed in the form  $u = \partial_t v$  with  $v$  the forward solution to  $\square v = f$ . Via the one dimensional reduction applied to  $v$ ,  $\nabla v$ ,  $\Omega v$ ,  $Sv$  and  $(t\partial_i + x_i\partial_t)v$  we obtain

$$|v| + |\nabla v| + |Sv| + |\Omega v| + \sum_i |(t\partial_i + x_i\partial_t)v| \lesssim \frac{1}{t \langle t - r \rangle}$$

where the main contribution comes from the cone. The above left hand side dominates  $\langle t - r \rangle \partial_t v$ ; therefore the proof of the lemma is complete.  $\square$

$\square$

**3.11. The fourth (and final) decay bound.** Here we reiterate one last time to remove the logarithms in (3.50) and establish the final bound, which concludes the proof of the Theorem.

**Lemma 3.21.** *The following decay estimate holds:*

$$(3.56) \quad |u_{\leq m}| \lesssim C_4 \frac{1}{t \langle t-r \rangle^2}, \quad |\nabla u_{\leq m}| \lesssim C_4 \frac{1}{\langle r \rangle \langle t-r \rangle^3}$$

where

$$C_4 = \|u_{\leq m+n}\|_{LE^1} + \sup_T \sum_{R,U} T^2 R^{\frac{1}{2}} U^{\frac{1}{2}} \|f_{\leq m+n}\|_{L^2(C_T^{R,U})} + T R^{\frac{3}{2}} U^{\frac{3}{2}} \|\nabla f_{\leq m+n}\|_{L^2(C_T^{R,U})}.$$

*Proof.* The argument is similar to the previous step, but with some extra care in order to avoid the logarithmic losses. The main goal is again to obtain a pointwise bound for  $u_{\leq m}$  which coincides with (3.50) near the cone,

$$(3.57) \quad |u_{\leq m}| \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^2} C_4$$

after which the full estimate (3.50) follows from Propositions 3.16, 3.15.

The pointwise bound for  $f_{\leq m}$  still has the form

$$|f_{\leq m}| \lesssim \frac{1}{t^2 \langle r \rangle^2 \langle t-r \rangle} C_4,$$

but now in addition we have dyadic summability with respect to  $R$  and  $U$ . Also by (3.50) we have

$$|u_{\leq m+6}| \lesssim \frac{\log^3 \langle t-r \rangle}{t \langle t-r \rangle^2} C_4, \quad |\nabla u_{\leq m+6}| \lesssim \frac{\log^3 \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^3} C_4.$$

We split  $G_{\leq m} = G_{\leq m}^1 + G_{\leq m}^2$  exactly as before.

For  $G_{\leq m}^1$  we use the one dimensional reduction, based on the bound

$$|G_{\leq m}^1| \lesssim |f_{\leq m}| + \frac{\log^3 \langle t-r \rangle}{\langle r \rangle^3 \langle t-r \rangle^3} C_4,$$

which gives the pointwise bound (3.57) for  $u_{\leq m}^1$ . The dyadic summability for  $f$  causes the absence of logarithms in the bound for the contribution of  $f$ . The contribution of the second term is of the order of

$$\frac{\log^3 \langle t-r \rangle}{\langle r \rangle \langle t-r \rangle^4}$$

where we have a full extra power of  $\langle t-r \rangle$  available to control the logarithms.

Finally, the contribution of  $G_{\leq m}^2$  is controlled by Lemma 3.20.  $\square$

#### 4. PERTURBATIONS OF KERR SPACETIMES

We now present an application of Theorem 1.5 to general relativity. We are able to recover Price's Law not only for Schwarzschild and Kerr spacetimes, but also for certain small, time-dependent perturbations thereof. We begin by presenting the results obtained in [45] and [66] for the Schwarzschild and Kerr metrics. We continue in 4.2 with a proof of stationary local energy decay estimates for perturbations of Schwarzschild; while the perturbations are required to be small, no decay to Schwarzschild is assumed. This result applies as well to small perturbations of

Kerr with small angular momentum. Finally, in 4.3 we establish weak local energy decay estimates for small perturbations of Kerr; here, we essentially require a  $t^{-1-}$  decay rate of the perturbed metric to Kerr.

**4.1. The Schwarzschild and Kerr metrics.** The Kerr metric in Boyer-Lindquist coordinates is given by

$$ds^2 = g_{tt}dt^2 + g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2$$

where  $t \in \mathbb{R}$ ,  $r > 0$ ,  $(\phi, \theta)$  are the spherical coordinates on  $\mathbb{S}^2$  and

$$\begin{aligned} g_{tt} &= -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2}, & g_{t\phi} &= -2a \frac{Mr \sin^2 \theta}{\rho^2}, & g_{rr} &= \frac{\rho^2}{\Delta} \\ g_{\phi\phi} &= \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta, & g_{\theta\theta} &= \rho^2 \end{aligned}$$

with

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta.$$

Here  $M$  represents the mass of the black hole, and  $aM$  its angular momentum.

A straightforward computation gives us the inverse of the metric:

$$\begin{aligned} g^{tt} &= -\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Delta}, & g^{t\phi} &= -a \frac{2Mr}{\rho^2 \Delta}, & g^{rr} &= \frac{\Delta}{\rho^2}, \\ g^{\phi\phi} &= \frac{\Delta - a^2 \sin^2 \theta}{\rho^2 \Delta \sin^2 \theta}, & g^{\theta\theta} &= \frac{1}{\rho^2}. \end{aligned}$$

The case  $a = 0$  corresponds to the Schwarzschild space-time. One can view  $M$  as a scaling parameter, and  $a$  scales in the same way as  $M$ . Thus  $M/a$  is a dimensionless parameter. We shall subsequently assume that  $a$  is small  $a/M \ll 1$ , so that the Kerr metric is a small perturbation of the Schwarzschild metric. One could also set  $M = 1$  by scaling, but we prefer to keep  $M$  in our formulas. We let  $g_{\mathbf{S}}$ ,  $g_{\mathbf{K}}$  denote the Schwarzschild, respectively Kerr metric, and  $\square_{\mathbf{S}}$ ,  $\square_{\mathbf{K}}$  denote the associated d'Alembertians.

In the above coordinates the Kerr metric has singularities at  $r = 0$  on the equator  $\theta = \pi/2$  and at the roots of  $\Delta$ , namely  $r_{\pm} = M \pm \sqrt{M^2 - a^2}$ . The singularity at  $r = r_+$  is just a coordinate singularity, and corresponds to the event horizon. The singularity at  $r = r_-$  is also a coordinate singularity; for a further discussion of its nature, which is not relevant for our results, we refer the reader to [11, 30]. To remove the singularities at  $r = r_{\pm}$  we introduce functions  $r^*$ ,  $v_+$  and  $\phi_+$  so that (see [30])

$$dr^* = (r^2 + a^2)\Delta^{-1}dr, \quad dv_+ = dt + dr^*, \quad d\phi_+ = d\phi + a\Delta^{-1}dr.$$

We call  $v_+$  the advanced time coordinate. The metric then takes the Eddington-Finkelstein form

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\rho^2}\right)dv_+^2 + 2drdv_+ - 4a\rho^{-2}Mr \sin^2 \theta dv_+ d\phi_+ - 2a \sin^2 \theta dr d\phi_+ \\ &\quad + \rho^2 d\theta^2 + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] \sin^2 \theta d\phi_+^2 \end{aligned}$$

which is smooth and nondegenerate across the event horizon up to but not including  $r = 0$ .

In order to talk about perturbations of Kerr we need to settle on a suitable coordinate frame. The Boyer-Lindquist coordinates are convenient at spatial infinity but not near the event horizon while the Eddington-Finkelstein coordinates are

convenient near the event horizon but not at spatial infinity. To combine the two we replace the  $(t, \phi)$  coordinates with  $(\tilde{t}, \tilde{\phi})$  as follows.

As in [45] and [66], we define

$$\tilde{t} = v_+ - \mu(r)$$

where  $\mu$  is a smooth function of  $r$ . In the  $(\tilde{t}, r, \phi_+, \theta)$  coordinates the metric has the form

$$\begin{aligned} ds^2 = & \left(1 - \frac{2Mr}{\rho^2}\right) d\tilde{t}^2 + 2 \left(1 - \left(1 - \frac{2Mr}{\rho^2}\right) \mu'(r)\right) d\tilde{t} dr \\ & - 4a\rho^{-2} Mr \sin^2 \theta d\tilde{t} d\phi_+ + \left(2\mu'(r) - \left(1 - \frac{2Mr}{\rho^2}\right) (\mu'(r))^2\right) dr^2 \\ & - 2a\theta(1 + 2\rho^{-2} Mr \mu'(r)) \sin^2 \theta dr d\phi_+ + \rho^2 d\theta^2 \\ & + \rho^{-2} [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] \sin^2 \theta d\phi_+^2. \end{aligned}$$

On the function  $\mu$  we impose the following two conditions:

- (i)  $\mu(r) \geq r^*$  for  $r > 2M$ , with equality for  $r > 5M/2$ .
- (ii) The surfaces  $\tilde{t} = \text{const}$  are space-like, i.e.

$$\mu'(r) > 0, \quad 2 - \left(1 - \frac{2Mr}{\rho^2}\right) \mu'(r) > 0.$$

For convenience we also introduce

$$\tilde{\phi} = \zeta(r) \phi_+ + (1 - \zeta(r)) \phi$$

where  $\zeta$  is a cutoff function supported near the event horizon and work in the  $(\tilde{t}, r, \tilde{\phi}, \theta)$  coordinates which are identical to  $(t, r, \phi, \theta)$  outside of a small neighborhood of the event horizon.

Given  $r_- < r_e < r_+$  we consider the Kerr metric and the corresponding wave equation

$$(4.1) \quad \square_{\mathbf{K}} u = f$$

in the cylindrical region

$$(4.2) \quad \mathcal{M} = \{\tilde{t} \geq 0, r \geq r_e\}$$

with initial data on the space-like surface

$$(4.3) \quad \Sigma^- = \mathcal{M} \cap \{\tilde{t} = 0\}.$$

The lateral boundary of  $\mathcal{M}$ ,

$$(4.4) \quad \Sigma^+ = \mathcal{M} \cap \{r = r_e\}, \quad \Sigma_{[\tilde{t}_0, \tilde{t}_1]}^+ = \Sigma^+ \cap \{\tilde{t}_0 \leq \tilde{t} \leq \tilde{t}_1\}$$

is also space-like and can be thought of as the exit surface for all waves which cross the event horizon. This places us in the context of Case B in section 1.1. The choice of  $r_e$  is not important; for convenience one may simply use  $r_e = M$  for all Kerr metrics with  $a/M \ll 1$  and small perturbations thereof.

A main difficulty in proving local energy decay in Schwarzschild/Kerr space-times is due to the presence of trapped rays (null geodesics). In the Schwarzschild case, this occurs on the photon sphere  $\{r = 3M\}$ . Consequently the local energy bounds have a loss at  $r = 3M$ . To localize there we use a smooth cutoff function  $\chi_{ps}(r)$  which is supported in a small neighborhood of  $3M$  and which equals 1 near

$3M$ . By  $\tilde{\chi}_{ps}(r)$  we denote a smooth cutoff that equals 1 on the support of  $\chi_{ps}$ . Then we define suitable modifications of the  $LE^1$ , respectively  $LE^*$  norms by

$$\begin{aligned}\|u\|_{LE^1_{\mathbb{S}}} &= \|\partial_r u\|_{LE} + \|(1 - \frac{3M}{r})\nabla u\|_{LE} + \|r^{-1}u\|_{LE} \\ \|f\|_{LE^*_{\mathbb{S}}} &= \|(1 - \chi_{ps})f\|_{LE^*} + \|\chi_{ps}f\|_{H^1 + (1 - \frac{3M}{r})L^2}\end{aligned}$$

With these notations, the local energy decay estimates established in [45] have the form

**Theorem 4.1.** *Let  $u$  solve  $\square_{\mathbb{S}}u = f$  in  $\mathcal{M}$ . Then*

$$(4.5) \quad \|u\|_{LE^1_{\mathbb{S}}} + \sup_{\tilde{v} \geq 0} \|\nabla u(\tilde{t})\|_{L^2} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{LE^*_{\mathbb{S}}}.$$

As written there is a loss of one derivative at  $r = 3M$ . This can be improved to an  $\epsilon$  loss, or even to a logarithmic loss, see [45], but that is not so relevant for our purpose here.

In the case of Kerr, the trapped rays are no longer localized on a sphere. However, if  $a/M \ll 1$  then they are close to the sphere  $\{r = 3M\}$ .

We now recall the setup and results from [66] for the Kerr spacetime. Let  $\tau, \xi, \Phi$  and  $\Theta$  be the Fourier variables corresponding to  $t, r, \phi$  and  $\theta$ , and

$$\begin{aligned}p_{\mathbf{K}}(r, \theta, \tau, \xi, \Phi, \Theta) &= g^{tt}\tau^2 + 2g^{t\phi}\tau\Phi + g^{\phi\phi}\Phi^2 + g^{rr}\xi^2 + g^{\theta\theta}\Theta^2 \\ &= g^{tt}(\tau - \tau_1(r, \theta, \xi, \Phi, \Theta))(\tau - \tau_2(r, \theta, \xi, \Phi, \Theta))\end{aligned}$$

be the principal symbol of  $\square_{\mathbf{K}}$ . Here  $\tau_1, \tau_2$  are real distinct smooth 1-homogeneous symbols. It is known that all trapped null geodesics in  $r > r_+$  satisfy

$$(4.6) \quad R_a(r, \tau, \Phi) = 0$$

where

$$R_a(r, \tau, \Phi) = (r^2 + a^2)(r^3 - 3Mr^2 + a^2r + a^2M)\tau^2 - 2aM(r^2 - a^2)\tau\Phi - a^2(r - M)\Phi^2.$$

Let  $r_a(\tau, \Phi)$  be root of (4.6) near  $r = 3M$ , which can be shown to exist and be unique for small  $a$ . Then define the symbols

$$c_i(r, \theta, \xi, \Phi, \Theta) = r - r_a(\tau_i, \Phi), \quad i = 1, 2$$

and the associated space-time norms:

$$\begin{aligned}\|u\|_{L^2_{c_i}}^2 &= \|c_i(D, x)u\|_{L^2}^2 + \|u\|_{H^{-1}}^2 \\ \|g\|_{c_i L^2}^2 &= \inf_{c_i(x, D)g_1 + g_2 = g} (\|g_1\|_{L^2}^2 + \|g_2\|_{H^1}^2).\end{aligned}$$

The replacements of the  $LE$  and  $LE^*$  norms are

$$\begin{aligned}\|u\|_{LE^1_{\mathbf{K}}} &= \|\chi_{ps}(D_t - \tau_2(D, x))c_1(D, x)\chi_{ps}u\|_{L^2}^2 \\ &\quad + \|\chi_{ps}(D_t - \tau_1(D, x))c_2(D, x)\chi_{ps}u\|_{L^2}^2 \\ &\quad + \|\partial_r u\|_{LE} + \|(1 - \chi^2)\nabla u\|_{LE} + \|r^{-1}u\|_{LE} \\ \|f\|_{LE^*_{\mathbf{K}}} &= \|(1 - \chi_{ps})f\|_{LE^*} + \|\chi_{ps}f\|_{c_1 L^2 \cap c_2 L^2}.\end{aligned}$$

Then the main result in [66] asserts that

**Theorem 4.2.** *Let  $u$  solve  $\square_{\mathbf{K}}u = f$  in  $\mathcal{M}$ . Then*

$$(4.7) \quad \|u\|_{LE^1_{\mathbf{K}}} + \sup_{\tilde{v} \geq 0} \|\nabla u(\tilde{t})\|_{L^2} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{LE^*_{\mathbf{K}}}.$$

This is the direct counterpart of (4.5), which corresponds to  $\tau_1 = -\tau_2$  and  $c_1 = c_2 = r - 3M$ . Again, the loss of one derivative can be improved to an  $\epsilon$  loss, or even to a logarithmic loss, see [67].

**4.2. Stationary local energy decay for small perturbations of Schwarzschild.** Here we consider a Lorentzian metric  $g$  in  $\mathcal{M}$  which is a small perturbation of Schwarzschild expressed in the  $(\tilde{t}, r, \phi, \theta)$  coordinates. Our main result is as follows:

**Theorem 4.3.** *Let  $g$  be a Lorentzian metric on  $\mathcal{M}$  and  $u$  a smooth function in  $\mathcal{M}$ .*

*a) Let  $\chi_{ps}$  be a smooth cutoff function which selects a small neighbourhood of the photon sphere  $\{r = 3M\}$ . If*

$$(4.8) \quad |\partial_\alpha [g_{\mu\nu} - (g_K)_{\mu\nu}]| \lesssim \epsilon \langle r \rangle^{-|\alpha|}, \quad 0 \leq |\alpha| \leq 1$$

*for a small enough  $\epsilon$ , then the stationary local energy bound holds for  $u$ :*

$$(4.9) \quad \|u\|_{LE^1[\tilde{t}_0, \tilde{t}_1]} \lesssim \|\nabla u(\tilde{t}_0)\|_{L^2} + \|\nabla u(\tilde{t}_1)\|_{L^2} + \|\square_g u\|_{LE^*[\tilde{t}_0, \tilde{t}_1]} + \|\chi_{ps} \partial_{\tilde{t}} u\|_{LE[\tilde{t}_0, \tilde{t}_1]}.$$

*b) If in addition*

$$(4.10) \quad |\partial^\alpha g^{\mu\nu}| \lesssim r^{-1-}, \quad 1 \leq |\alpha| \leq k+1$$

*then (1.11) also holds.*

*Proof.* a) The argument is based on the computation in [45], which only requires the use of vector fields. We will first prove the estimate in the Schwarzschild case, but do it in such a way so that the transition to  $g$  is perturbative.

Let  $X$  be a differential operator

$$(4.11) \quad X = b(r)\partial_r + c(r)\partial_{\tilde{t}} + q(r)$$

for some smooth functions  $b, c, q : [r_e, \infty) \rightarrow \mathbb{R}$  with  $c$  constant outside a compact region and  $b, q$  satisfying

$$(4.12) \quad \begin{aligned} |\partial_r^\alpha b| &\leq c_\alpha r^{-|\alpha|} \\ |\partial_r^\alpha q| &\leq c_\alpha r^{-1-|\alpha|}. \end{aligned}$$

Let  $\mathcal{M}_{[\tilde{t}_0, \tilde{t}_1]} = \{\tilde{t}_0 < \tilde{t} < \tilde{t}_1, r > r_e\}$ , and let  $dV_S = r^2 dr d\tilde{t} d\omega$  denote the Schwarzschild induced measure. It was shown in [45] that one can find  $X$  as above so that

$$(4.13) \quad \int_{\tilde{t}_0}^{\tilde{t}_1} Q^S(\tilde{t}) d\tilde{t} = - \int_{\mathcal{M}_{[\tilde{t}_0, \tilde{t}_1]}} \square_S u \cdot Xu \, dV_S - BDR^S[u] \Big|_{\tilde{t}=\tilde{t}_0}^{\tilde{t}=\tilde{t}_1} - BDR^S[u] \Big|_{r=r_e}$$

with

$$\int_{\tilde{t}_0}^{\tilde{t}_1} Q^S(\tilde{t}) d\tilde{t} \gtrsim \|u\|_{LE_{S,w}^1[\tilde{t}_0, \tilde{t}_1]}^2$$

where

$$\|u\|_{LE_{S,w}^1} = \|(r - 3M)\chi \nabla u\|_{L^2}^2 + \|r^{-2} \partial_r u\|_{L^2}^2 + \|r^{-1} u\|_{L^2}^2 + \|(1 - \chi^2) \nabla u\|_{L^2}^2$$

and the boundary terms satisfy

$$\begin{aligned} BDR^S[u] \Big|_{\tilde{t}=\tilde{t}_i} &\approx \|\nabla u(\tilde{t}_i)\|_{L^2}^2, \quad i = 1, 2 \\ BDR^S[u] \Big|_{r=r_e} &\approx \|u\|_{H^1(\Sigma_{[\tilde{t}_0, \tilde{t}_1]}^+)}^2. \end{aligned}$$



Comparing the  $LE_{\mathbf{S},w}^1$  norm we obtain from this computation with the  $LE^1$  norm which we need, one sees that two improvements are necessary, one near  $r = 3M$  and another for large  $r$ .

The improvement for large  $r$  is a consequence of the fact that for large  $r$  one can view the Schwarzschild metric as a small perturbation of the Minkowski metric. Precisely, from the estimate [49](2.3) (see also [48]) we have for large  $R$

$$(4.14) \quad \|\chi_{>R}u\|_{LE^1[\tilde{t}_0,\tilde{t}_1]} \lesssim \|u\|_{LE_{\mathbf{S},w}^1[\tilde{t}_0,\tilde{t}_1]} + \|\chi_{>R}\square_{\mathbf{S}}u\|_{LE[\tilde{t}_0,\tilde{t}_1]} + \|\nabla u(\tilde{t}_0)\|_{L^2}.$$

The similar bound for the metric  $g$  is also valid.

To gain the improvement for  $r$  close to  $3M$  we add a Lagrangian correction term. Precisely, a direct integration by parts yields

$$\int_{\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1]}} \square_{\mathbf{S}}u \cdot \chi_{ps}^2 u \, dV_{\mathbf{S}} = \int_{\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1]}} \chi_{ps}^2 g_{\mathbf{S}}^{\mu\nu} \partial_{\mu}u \partial_{\nu}u + (\square_{\mathbf{S}}\chi_{ps}^2)u^2 \, dV_{\mathbf{S}} + \int_{\mathcal{M}_{\tilde{t}}} \chi_{ps}^2 g_{\mathbf{S}}^{00} u \partial_t u \, dx \Big|_{\tilde{t}=\tilde{t}_0}^{\tilde{t}=\tilde{t}_1}.$$

Since  $\partial_t$  is timelike in the support of  $\chi_{ps}$ , we can write the pointwise bound

$$|\chi_{ps}\nabla u|^2 \lesssim \chi_{ps}^2 g_{\mathbf{S}}^{\mu\nu} \partial_{\mu}u \partial_{\nu}u + C|\chi_{ps}\partial_t u|^2$$

for some large constant  $C$ . Then the previous identity yields

$$(4.15) \quad \|\chi_{ps}\nabla u\|_{L^2(\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1])}}^2 \lesssim \int_{\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1]}} \square_{\mathbf{S}}u \cdot \chi_{ps}^2 u \, dV_{\mathbf{S}} + C\|\chi_{ps}\partial_t u\|_{L^2(\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1])}}^2 + \sum_{i=1,2} \|\nabla u(\tilde{t}_i)\|_{L^2}.$$

Combining (4.13), (4.14) and (4.15) we obtain

$$(4.16) \quad \begin{aligned} & \|u\|_{LE^1(\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1])}}^2 + \|u\|_{H^1(\Sigma_{[\tilde{t}_0,\tilde{t}_1]}^+)}^2 + \|\nabla u(\tilde{t}_1)\|_{L^2}^2 \lesssim - \int_{\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1]}} \square_{\mathbf{S}}u \cdot X_1 u \, dV_{\mathbf{S}} \\ & + \|\nabla u(\tilde{t}_0)\|_{L^2}^2 + C\|\chi_{ps}\partial_t u\|_{L^2(\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1])}}^2 + \|\chi_{>R}\square_{\mathbf{S}}u\|_{LE^*(\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1])}}^2 \end{aligned}$$

where

$$X_1 = X + \delta\chi_{ps}^2(r)$$

with a fixed small constant  $\delta$ . At this point, the desired conclusion (4.9) for the Schwarzschild metric follows if we estimate the integral term by  $\|\square_{\mathbf{S}}u\|_{LE^*}\|u\|_{LE^1}$ .

It remains to show that a similar estimate holds with  $\square_{\mathbf{S}}$  replaced by  $\square_g$ . This substitution is easily made in the last term on the right by performing a similar substitution in (4.14). Consider now the difference in the integral term,

$$D = \int_{\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1]}} (\square_{\mathbf{S}} - \square_g)u \cdot X_1 u \, dV_{\mathbf{S}}.$$

To estimate this we use the bound (4.8) to write

$$\square_{\mathbf{S}} - \square_g = \partial_{\mu}(g_{\mathbf{S}}^{\mu\nu} - g^{ij})\partial_{\nu} + O(\epsilon r^{-1-})\nabla.$$

Then we integrate by parts in a standard manner. Using also Hardy type inequalities we obtain

$$|D| \lesssim \epsilon \left( \sum_{i=1,2} \|\nabla u(\tilde{t}_i)\|_{L^2}^2 + \|u\|_{H^1(\Sigma_{[\tilde{t}_0,\tilde{t}_1]}^+)}^2 + \|u\|_{LE^1(\mathcal{M}_{[\tilde{t}_0,\tilde{t}_1])}}^2 \right).$$

Hence (4.16) for  $\square_g$  follows, and the proof of the stationary local energy bound (4.9) is concluded.

b) The proof follows closely that of Theorem 4.5 in [66], but for the sake of completeness we include it here. The result will follow by induction on  $k$ . The case  $k = 0$  is part (a) of the theorem. We will prove the case  $k = 1$ , and the rest follows in a similar manner.

We need to estimate  $\|\nabla^2 u\|_{LE[\tilde{t}_0, \tilde{t}_1]}$ . We already control  $\|\nabla \partial_{\tilde{t}} u\|_{LE[\tilde{t}_0, \tilde{t}_1]}$ , therefore it remains to estimate the second order spatial derivatives. We write

$$\square_g u = L_1 \partial_{\tilde{t}} u + L_2 u$$

where  $L_1$  is a first order operator and  $L_2$  is a purely spatial second order operator. Then we have

$$\|L_2 u\|_{LE[\tilde{t}_0, \tilde{t}_1]} \lesssim \|\square_g u\|_{LE[\tilde{t}_0, \tilde{t}_1]} + \|L_1 \partial_{\tilde{t}} u\|_{LE[\tilde{t}_0, \tilde{t}_1]}$$

which is favorable where  $L_2$  is elliptic. But  $L_2$  is elliptic wherever  $\partial_{\tilde{t}}$  is time-like. Since  $g$  is a small perturbation of  $g_S$ , this happens everywhere outside a small neighbourhood of  $r = 2M$ . Thus we have the elliptic estimate

$$(4.17) \quad \|\chi_{out} \nabla^2 u\|_{LE[\tilde{t}_0, \tilde{t}_1]} \lesssim \|\square_g u\|_{LE[\tilde{t}_0, \tilde{t}_1]} + \|\nabla \partial_{\tilde{t}} u\|_{LE[\tilde{t}_0, \tilde{t}_1]} + \|\nabla u\|_{LE[\tilde{t}_0, \tilde{t}_1]}$$

where  $\chi_{out}$  selects the region  $\{r > 2M + \delta\}$ .

It remains to estimate  $\|\nabla^2 \chi_{eh} u\|_{L^2}$  where  $\chi_{eh}$  is a smooth cutoff function which selects the region  $\{r < 2M + 2\delta\}$  near the event horizon. The function  $v = \chi_{eh} u$  solves the equation

$$\square_g v = h := \chi_{eh} f + [\square_g, \chi_{eh}]$$

where the second term on the right is controlled in  $H^1$  via (4.17). Hence the conclusion of part (b) of the Proposition would follow from the following

**Lemma 4.4.** *Let  $\mathcal{M}_{eh} = \{r_e < r < 2M + 3\delta, \tilde{t} \geq 0\}$  with a fixed small  $\delta$ . Let  $g$  be an  $O(\epsilon)$  perturbation of  $g_S$  in  $C^{m+1}(\mathcal{M}_{eh})$  with  $\epsilon$  sufficiently small. Then for all functions  $v$  with support in  $\{r < 2M + 3\delta\}$  we have*

$$(4.18) \quad \|\nabla v\|_{H^m(\mathcal{M}_{eh})} \lesssim \|\nabla v(0)\|_{H^m} + \|\square_g v\|_{H^m(\mathcal{M}_{eh})}$$

*The similar estimate holds in any interval  $[\tilde{t}_0, \tilde{t}_1]$ .*

*Proof.* This is an estimate which is localized near the event horizon, and we will prove it taking advantage of the red shift effect. In microlocal terms, the red shift effect is equivalent to exponential energy decay along the light rays which stay on the event horizon, and small perturbations thereof. But for this estimate, these are all light rays of interest. All others exit the domain  $\mathcal{M}_{eh}$  in a finite time.

We begin with a simplification. If  $\epsilon$  is small enough then for  $r < 2M - \delta$  the  $r$  spheres are uniformly time-like, therefore we can use standard local energy estimates for the wave equation to reduce the problem to the case when  $r_e = 2M - 2\delta$ .

For  $m = 0$  the above bound follows from part (a) of the Proposition. For  $m = 1$  we commute  $\square_g$  with the vector fields  $\partial_{\tilde{t}}$ ,  $\Omega$  and  $\partial_r$ . We have

$$[\partial_{\tilde{t}}, \square_g] = O(\epsilon)Q_2, \quad [\Omega, \square_g] = O(\epsilon)Q_2$$

for some second order partial differential operator  $Q_2$  with bounded coefficients. Hence applying (4.18) with  $m = 0$  to  $\partial_{\tilde{t}} v$  and  $\Omega v$  we obtain

$$(4.19) \quad \|\Omega v\|_{H^1(\mathcal{M}_{eh})} + \|\partial_{\tilde{t}} v\|_{H^1(\mathcal{M}_{eh})} \lesssim \|h\|_{H^1(\mathcal{M}_{eh})} + \epsilon \|v\|_{H^2(\mathcal{M}_{eh})}.$$

We still need to bound  $\partial_r v$ . For that we compute the commutator

$$(4.20) \quad [\square_g, \partial_r] = -(\partial_r g_S^{rr}) \partial_r^2 + O(\epsilon)Q_2 + N_2$$

where  $N_2$  stands for a second order operator with no  $\partial_r^2$  terms. The key observation here is that  $\gamma = \partial_r g_{\mathbf{S}}^{rr} > 0$  near  $r = 2M$ . We can now write

$$(\square_g - \gamma_1 X) \partial_r v = \partial_r h + (O(\epsilon)Q_2 + N_2)v, \quad \gamma_1 > 0$$

with  $N_2$  as above and most importantly, a positive coefficient  $\gamma_1$ . We recall here that  $X$  looks like  $-\partial_r$  near the event horizon. Because of this the operator

$$B = \square_g - \gamma_1 X$$

satisfies the same estimate (4.9) as  $\square_g$  for functions supported near the event horizon. To see this it suffices to examine (4.16) with  $\square_{\mathbf{S}}$  replaced by  $\square_g$ . Since  $X = X_1$  near the event horizon, it follows that the contribution of  $\gamma_1 X$  has the right sign and can be discarded. Hence we obtain

$$(4.21) \quad \begin{aligned} \|\partial_r v\|_{H^1(\mathcal{M}_{eh})} &\lesssim \|h\|_{H^1(\mathcal{M}_{eh})} + \epsilon \|v\|_{H^2(\mathcal{M}_{eh})} \\ &\quad + \|\Omega v\|_{H^1(\mathcal{M}_{eh})} + \|\partial_t v\|_{H^1(\mathcal{M}_{eh})} + \|v\|_{H^1(\mathcal{M}_{eh})} \end{aligned}$$

where the last three terms account for the effect of  $N_2$ . Combining the bounds (4.19) and (4.21) we obtain

$$\|v\|_{H^2(\mathcal{M}_{eh})} \lesssim \|h\|_{H^1(\mathcal{M}_{eh})} + \epsilon \|v\|_{H^2(\mathcal{M}_{eh})} + \|v\|_{H^1(\mathcal{M}_{eh})}.$$

If  $\epsilon$  is sufficiently small then the conclusion (4.18) follows for  $m = 1$ . The argument for  $m > 1$  is similar.  $\square$

$\square$

**4.3. Local energy decay for small perturbations of Kerr.** Here we consider small perturbations of a Kerr space-time with small angular momentum,  $|a| \ll M$ . Our main result is as follows:

**Theorem 4.5.** *Let  $g$  be a Lorentzian metric on  $\mathcal{M}$  and  $u$  a function in  $\mathcal{M}$  solving  $\square_g u = f$ .*

*a) Assume that  $g$  satisfies (4.8) and decays to  $g_{\mathbf{K}}$  near the photon sphere,*

$$(4.22) \quad \chi_{ps} |\partial_\alpha [g_{\mu\nu} - (g_{\mathbf{K}})_{\mu\nu}]| \leq c_\alpha(\tilde{t}), \quad 0 \leq |\alpha| \leq 1$$

*where  $c_\alpha \in L_t^1$  (in particular, we can take  $c_\alpha = \langle \tilde{t} \rangle^{-1-}$ ). Then the weak local energy estimate holds:*

$$(4.23) \quad \|u\|_{H^1(\Sigma_R^+)}^2 + \sup_{\tilde{t} \geq 0} \|\nabla u(\tilde{t})\|_{L^2}^2 + \|u\|_{LE_{\mathbf{K}}^1}^2 \lesssim \|\nabla u(0)\|_{L^2}^2 + \|f\|_{LE_{\mathbf{K}}^*}^2.$$

*b) Assume in addition that (4.10) holds. Then*

$$(4.24) \quad \|u\|_{H^k(\Sigma_R^+)}^2 + \sup_{\tilde{t} \geq 0} \|\nabla u(\tilde{t})\|_{H^k}^2 + \|u\|_{LE_{\mathbf{K}}^{1,k}}^2 \lesssim \|\nabla u(0)\|_{H^k}^2 + \|f\|_{LE_{\mathbf{K}}^{*,k}}^2,$$

*and thus (1.9) holds.*

*Proof.* On any fixed compact time interval we have uniform energy estimates. Eliminating a compact time interval, we can assume without any restriction in generality that the integrability condition on  $c(\tilde{t})$  is strengthened to

$$(4.25) \quad \int_0^\infty c(\tilde{t}) d\tilde{t} \lesssim \epsilon, \quad |c(\tilde{t})| \lesssim \epsilon$$

where (4.8) was also used.

a) The proof of (4.23) is similar to the proof of Theorem 4.3, but it requires the use of pseudodifferential operators. Let us start by recalling the idea behind

the proof of Theorem 4.2. It is shown in [66] that there exists a pseudodifferential operator  $S_1$  of order 1 that satisfies the following:

- a)  $S_1$  is a differential operator in  $\tilde{t}$  of order 1.
- b)  $S_1 = X + \chi_{ps}s^w\chi_{ps}$ , where  $X$  is defined as in (4.11) and  $s \in S^1$ .
- c) Let  $\mathcal{M}_{[0,\tilde{t}_0]} = \{0 < \tilde{t} < \tilde{t}_0, r > r_e\}$  and  $dV_{\mathbf{K}} = \rho^2 dr d\tilde{t} d\omega$  denote the Kerr induced measure. Then one has

$$(4.26) \quad \int_0^{\tilde{t}_0} Q^{\mathbf{K}}(\tilde{t}) d\tilde{t} = - \int_{\mathcal{M}_{[0,\tilde{t}_0]}} (\square_{\mathbf{K}} u)(S_1 u) dV_{\mathbf{K}} - BDR^{\mathbf{K}}[u] \Big|_{\tilde{t}=0}^{\tilde{t}=\tilde{t}_0} - BDR^{\mathbf{K}}[u] \Big|_{r=r_e}$$

with

$$\begin{aligned} \int_0^{\tilde{t}_0} Q^{\mathbf{K}}(\tilde{t}) d\tilde{t} &\gtrsim \|u\|_{LE_{\mathbf{K},w}^1[0,\tilde{t}_0]}^2 \\ \|u\|_{LE_{\mathbf{K},w}^1}^2 &= \|\chi_{ps}(D_{\tilde{t}} - \tau_2(D, x))c_1(D, x)\chi_{ps}u\|_{L^2}^2 \\ &\quad + \|\chi_{ps}(D_{\tilde{t}} - \tau_1(D, x))c_2(D, x)\chi_{ps}u\|_{L^2}^2 \\ &\quad + \|r^{-2}\partial_r u\|_{L^2}^2 + \|r^{-1}u\|_{L^2}^2 + \|(1 - \chi_{ps}^2)\nabla u\|_{L^2}^2 \end{aligned}$$

and the boundary terms satisfying

$$(4.27) \quad \begin{aligned} BDR^{\mathbf{K}}[u] \Big|_{\tilde{t}=\tilde{t}_i} &\approx \|\nabla u(\tilde{t}_i)\|_{L^2} \\ BDR^{\mathbf{K}}[u] \Big|_{r=r_e} &\approx \|u\|_{H^1(\Sigma_{[0,\tilde{t}_0]}^+)}^2 \end{aligned}$$

Note that conditions a) and b) guarantee that the boundary terms are well-defined after integrating by parts.

The same reasoning as in Theorem 4.3 leads to the counterpart of (4.16), namely

$$(4.28) \quad \begin{aligned} &\|u\|_{LE_{\mathbf{K}}^1(\mathcal{M}_{[0,\tilde{t}_0]})}^2 + \|\nabla u\|_{L^2(\Sigma_{[0,\tilde{t}_0]}^+)}^2 + \|\nabla u(\tilde{t}_1)\|_{L^2}^2 \lesssim \\ &- \int_{\mathcal{M}_{[0,\tilde{t}_0]}} \square_{\mathbf{K}} u \cdot S_1 u \, dV_{\mathbf{K}} + \|\nabla u(\tilde{t}_0)\|_{L^2}^2 + \|\chi_{>R} \square_{\mathbf{K}} u\|_{LE^*(\mathcal{M}_{[0,\tilde{t}_0]})}^2. \end{aligned}$$

Here we seek to replace the Kerr metric by  $g$ . As discussed in Theorem 4.3, the bound (4.14) holds as well for the metric  $g$ , therefore the last term is not an issue. Hence it remains to consider the difference

$$D = \int_{\mathcal{M}_{[0,\tilde{t}_0]}} (\square_{\mathbf{K}} - \square_g) u \cdot S_1 u \, dV_{\mathbf{K}}$$

We split  $S_1 = X_1 + S_{ps}$  where

$$X_1 = X - \chi_{ps}X\chi_{ps}, \quad S_{ps} = \chi_{ps}X\chi_{ps} + \chi_{ps}s^w\chi_{ps}.$$

Thus  $X_1$  is a first order differential operator which is supported away from the photon sphere. Correspondingly we split  $D = D_1 + D_{ps}$ . For  $D_1$ , integration by parts using (4.8) leads to

$$|D_1| \lesssim \epsilon(\|\nabla u(0)\|_{L^2}^2 + \|\nabla u(\tilde{t}_0)\|_{L^2}^2 + \|u\|_{H^1(\Sigma_{[0,\tilde{t}_0]}^+)}^2 + \|u\|_{LE_{\mathbf{K}}^1(\mathcal{M}_{[0,\tilde{t}_0]})}^2).$$

Here it is essential that the outcome of the integration by parts is supported away from the photon sphere  $\{r = 3M\}$ , where the  $LE_{\mathbf{K}}^1$  and  $LE^1$  norms are equivalent.

To estimate  $D_{ps}$  we need to use the stronger bound (4.22). Then near the photon sphere we can write

$$\square_{\mathbf{K}} - \square_g = \frac{1}{\sqrt{|g_{\mathbf{K}}|}} \partial_\mu \sqrt{|g_{\mathbf{K}}|} (g_{\mathbf{K}}^{\mu\nu} - g^{\mu\nu}) \partial_\nu + O(c(\tilde{t})) \nabla$$

where we also have  $g_{\mathbf{K}}^{\mu\nu} - g^{\mu\nu} = O(c(\tilde{t}))$  and  $\nabla(g_{\mathbf{K}}^{\mu\nu} - g^{\mu\nu}) = O(c(\tilde{t}))$ . Thus integrating by parts we obtain

$$|D_{ps}| \lesssim \int_{\mathcal{M}_{[0, \tilde{t}_0]}} \tilde{\chi}_{ps} c(\tilde{t}) (|\nabla u|^2 + |u|^2) dV_{\mathbf{K}} + c(0) \|\nabla u(0)\|_{L^2}^2 + c(\tilde{t}_0) \|\nabla u(\tilde{t}_0)\|_{L^2}^2.$$

By (4.25) it follows that

$$|D_{ps}| \lesssim \epsilon \left( \sup_{\tilde{t} \in [0, \tilde{t}_0]} \|\nabla u(\tilde{t})\|_{L^2}^2 + \|u\|_{LE_{\mathbf{K}}^1(\mathcal{M}_{[0, \tilde{t}_0]})}^2 \right).$$

Applying the bounds for  $D_1$  and  $D_{ps}$ , we complete the replacement of  $\square_{\mathbf{K}}$  by  $\square_g$  in (4.28), obtaining

$$(4.29) \quad \begin{aligned} & \|u\|_{LE_{\mathbf{K}}^1(\mathcal{M}_{[0, \tilde{t}_0]})}^2 + \|\nabla u\|_{L^2(\Sigma_{[0, \tilde{t}_0]}^+)}^2 + \|\nabla u(\tilde{t}_0)\|_{L^2}^2 \lesssim - \int_{\mathcal{M}_{[0, \tilde{t}_0]}} \square_g u \cdot S_1 u \, dV_{\mathbf{K}} \\ & + \|\nabla u(0)\|_{L^2}^2 + \|\chi_{>R} \square_g u\|_{LE^*(\mathcal{M}_{[0, \tilde{t}_0]})}^2 + \epsilon \sup_{\tilde{t} \in [0, \tilde{t}_0]} \|\nabla u(\tilde{t})\|_{L^2}^2 \end{aligned}$$

To conclude the proof of (4.23) we estimate the integral term in the last inequality by  $\|u\|_{LE_{\mathbf{K}}^1(\mathcal{M}_{[0, \tilde{t}_0]})} \|\square_g u\|_{LE_{\mathbf{K}}^*}$  and use Cauchy-Schwarz. The last term on the right is eliminated by taking the supremum in the resulting estimate over  $\tilde{t}_0 \in [0, \tilde{t}_1]$  for arbitrary  $\tilde{t}_1 > 0$ .

b) We prove the estimate (4.24) for  $k = 1$ ; the argument for larger  $k$  is identical. We begin by applying the estimate (4.23) to  $\partial_{\tilde{t}} u$ . Commuting  $\square_g$  with  $\partial_{\tilde{t}}$  we have

$$\square_g \partial_{\tilde{t}} u = \partial_{\tilde{t}} \square_g u + O(\epsilon r^{-1-}) [(1 - \chi_{ps}) Q_2 u + \chi_{ps} Q_1 u] + O(c(\tilde{t})) \chi_{ps} Q_2 u$$

where  $Q_1$  and  $Q_2$  stand for second order operators with bounded coefficients. Here we have used (4.8) for first derivatives of  $g$  away from the photon sphere, (4.22) for first derivatives of  $g$  near the photon sphere, and (4.10) for second order derivatives of  $g$ . We estimate the second term in  $\square_g \partial_{\tilde{t}} u$  in  $LE_{\mathbf{K}}^*$  and the third in  $L_{\tilde{t}}^1 L_x^2$ . This gives

$$\|\partial_{\tilde{t}} u\|_{LE_{\mathbf{K}}^1} \lesssim \epsilon \|u\|_{LE_{\mathbf{K}}^{1,1}} + \|u\|_{LE_{\mathbf{K}}^1} + \|\square_g u\|_{LE_{\mathbf{K}}^{*,1}}.$$

Away from the event horizon the vector field  $\partial_{\tilde{t}}$  is timelike, therefore, arguing as in the proof of Theorem 4.3(b), we can use an elliptic estimate to conclude that

$$\|\chi_{out} u\|_{LE_{\mathbf{K}}^{1,1}} \lesssim \epsilon \|u\|_{LE_{\mathbf{K}}^{1,1}} + \|u\|_{LE_{\mathbf{K}}^1} + \|\square_g u\|_{LE_{\mathbf{K}}^{*,1}}.$$

On the other hand, near the event horizon we use Lemma 4.4 to obtain

$$\|\chi_{eh} u\|_{H^2} \lesssim \|\chi_{out} u\|_{H^2} + \|\chi_{eh} \square_g u\|_{H^1} + \|\nabla u(0)\|_{H^1}$$

Combining the last three estimates we obtain (4.24) for  $k = 1$ .  $\square$

**4.4. Conclusion.** We can now prove Price’s law for certain perturbations of the Kerr spacetimes:

**Theorem 4.6.** *Let  $g$  be a Lorentzian metric close to  $g_K$  in the sense that it satisfies (4.8), (4.10) and (4.22). Let  $u$  solve (1.1) with smooth, compactly supported initial data and  $V = 0$ . Then (1.12) holds.*

*Proof.* This is an obvious consequence of Theorems 1.5, 4.3 and 4.5. □

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